YUMS-94-23, SWAT/65, hep-th/9503036

Holomorphic Yang-Mills Theory and Variation of the Donaldson Invariants

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Abstract

We study the path integrals of the holomorphic Yang-Mills theory on compact Kähler surface with $b_2^+ = 1$. Based on the results, we examine the correlation functions of the topological Yang-Mills theory and the corresponding Donaldson invariants as well as their transition formulas.

Feb, 1995

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1. Introduction

The Donaldson polynomial invariants are powerful tools for classifying the smooth structures of four-manifolds [1][2]. For a Riemann manifold with $b_2^+ \geq 3$, these polynomials are well-defined and metric independent, as long as it is generic.

On the other hand, for a manifold with $b_2^+ = 1$, the polynomials depend on metric in a very interesting way. This is due to the reducible anti-self-dual (ASD) connections which appear at finite points of the generic smooth path connecting two generic metrics. The first example was studied by Donaldson in his seminal paper on the failure of the h-cobordism conjecture [3]. His formula was further studied in detail by Friedman-Morgan [4], and generalized by Kotschick [5], Mong [6], and Kotschick-Morgan [7].

The topological Yang-Mills (TYM) theory proposed by Witten is a field theoretic interpretation of the Donaldson invariants [8]. The relation with the theory to the BRST quantization were much studied in [9][10][11][12][13][14].[1] It is also a new mathematical viewpoint on the Donaldson invariants due to the Atiyah-Jeffrey re-interpretation [16] of the Witten's approach as an infinite dimensional generalization of the Mathai-Quillen representative [17] of the equivariant Thom class. The TYM theory on compact Kähler surface was studied in some detail by the second author [18]. It was shown that the theory has two global fermionic symmetries.[2] The theory can be interpreted as an infinite dimensional version of a Dolbeault equivariant cohomological analogue of the Mathai-Quillen formalism [20][21]. Witten studied the Kähler case by twisting the $N = 2$ super-Yang-Mills theory and determined the Donaldson Polynomial invariants for Kähler surfaces with $b_2^+ \geq 3$ almost completely[22]. His approach gives yet another new perspective on the Donaldson theory since he related the theory to the infrared behavior of the $N = 1$ super-Yang-Mills theory.

On the other hand, to the authors' knowledge, no serious attempt has been made for the field theoretic study of the Donaldson polynomial invariants for manifolds with $b_2^+ = 1$.[3] The TYM theory may not be well-defined in the presence of the reducible

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[1] We refer to ref. [15] as a review and for further references.

[2] The first formulation of the TYM theory on Kähler surface was given by [19].

[3] In ref. [23], it was shown that the variation of the Donaldson invariants is related to the holomorphic anomaly.

anti-self-dual (ASD) connections. The reducible connections also contribute to the non-compactness of the space where the path integral is localized, due to the flat directions of the scalar potential. This makes the topological interpretation of the theory unclear [24]. Furthermore, the crucial perturbation of Witten [22], inducing the mass gap to the theory, is not applicable to this case[4].

In this paper, we study the $SU(2)$ Donaldson polynomial invariants of a simply connected compact Kähler surface X with the vanishing geometric genus $p_g(X) = 0$, i.e., $b_2^+(X) = 1$, based on the holomorphic Yang-Mills (HYM) theory [25]. The HYM theory was proposed by the second author to provide yet another field theoretic interpretation of the Donaldson invariants of Kähler surface, adopting the two dimensional construction of Witten [26]. It was shown that there exists a simple mapping to TYM theory on Kähler surface analoguos to the mapping from two-dimensional TYM theory to the physical YM theory. It turns out that the HYM theory is more useful for manifold with $p_g = 0$ [5]. A la HYM theory we will show that the path integral approach to the Donaldson theory is well-defined whatever properties the moduli space of ASD connections has [8].

The basic idea of the HYM theory is that the moduli space of ASD connections over Kähler surface is the symplectic quotients in the space $\mathcal{A}^{1,1}$ of all holomorphic connections. Classically, the action functional of the HYM theory is the norm squared of moment map. The partition function of the theory can be expressed by the values of the moment map at the critical points of the action functional. The spectrum of the critical points, as one varies the metric, depends on a certain chamber structure of the positive cone. By studying the partition function and some correlation functions in the small coupling limit, one can obtain the expectation values of certain topological observables of TYM theory. It turns out those expectation values are well-defined and depend only on the chamber structure of the metric.

The basic method of our calculation is the fixed point theorem of Witten [27]. Unfortunately, we could calculate only one of the two branches of the fixed points. The branch we calculate does not contain any information on the genuine diffeomorphism invariants.

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[4] In ref. [20], we have exploited the origin of the mass gap in terms of the Dolbeault model of the equivariant cohomology.

[5] The HYM theory was also used to determine the non-algebraic part of the Donaldson invariants of manifold with $H^{2,0} \neq 0$ [20].

While we were struggling with this problem, the Donaldson theory evolved into the new dimension by the fundamental works of Seiberg and Witten on the strong coupling properties of the $N = 2$ super-Yang-Mills theory [28][29]. By exploiting a dual Donaldson theory (Seiberg-Witten theory), Witten determined the invariants completely for Kähler surface with $b_2^+ \geq 3$[30]. This new method resolves all of the difficulties of the Donaldson theory as well as the TYM theory, such as the non-compactness of the moduli space of ASD connections. For a manifold with $b_2^+ = 1$, Witten's new approach has an additional complication comparing to the case with $b_2^+ \geq 3$[30]. The calculation we have done in this paper can be viewed as the path integral contributed from the generic points of the quantum moduli space (the complex u-plane in [30]) except the two singular points. We will argue that the remaining branch of the partition function of HYM theory may be obtained by the Seiberg-Witten invariants which corresponds to the path integral contributed from the two singular points in the quantum moduli space.

As a matter of fact, the expectation values of TYM theory are not identical with the corresponding Donaldson invariants. The difference is originated from the compactification of the moduli space ASD connections in the mathematical definition of the Donaldson invariants. In the field theoretic approach, both TYM and HYM theories do not refer to any compactification procedure. In any case, the path integrals are turned out to be well-defined though the moduli space is rarely compact. To recover the original Donaldson invariants some explicit prescription for incorporation of the compactification may be required. There are many reasons that the only difference between the two approaches is that the Donaldson theory has additional contributions due to reducible connections with lower instanton numbers. However, the additional part does not have any relevant information on the diffeomorphism class of the underlying manifolds.

We will interpret our results as the contributions of the original moduli space of ASD connections which is the top stratum of the compactified moduli space. Then, our results can be used to determine the transition formula of the Donaldson invariants. In some respects our results are more general than the known mathematical results. We also determine the non-analytic (non-polynomial) property of the invariants when the moduli space is singular. We believe that our formula gives a very clear and an intuitive understanding of the mechanism of the variation of the Donaldson invariants. This allows us to predict a rather detailed formula of the Donaldson invariants if the transition formula is known.

The invariants and the explicit transition formulas for some special cases were calculated in [31][32]. While we are preparing this paper the general transition formula appeared in term of some enumerative geometry of the Hilbert schemes[33][34][35]. The paper of Friedman and Qin [33] contains some explicit results on the transition formula. The paper of Ellingsrud and Göttsche [34]contains more general explicit results.[6]. Their mathematical picture on the variation of the moduli space is quite similar to our physical picture. At the moment, the invariants are calculated only for some special cases and no general explicit transition formula is known [31][32].

This paper is organized as follows. In Sect. 2, we review the Donaldson polynomial invariants, the TYM theory and the HYM theory. We show that the HYM theory is a natural field theoretic model for the Donaldson invariants of manifolds with $p_g = 0$. In Sect. 3, we give the path integral calculation of the HYM theory and obtain an effective theory. In Sect. 4, we examine the small coupling behavior of the partition function and expectation values of topological observables. Then, we study the variation of the path integral according to the variation of the metric. In Sect. 5, we obtain the expectation values in the TYM theory from the result of HYM theory. We argue that the unknown branch of the path integral corresponds to the Seiberg-Witten monopole invariants. After a brief discussion of some properties of the monopole invariants, we show that the expectation values of TYM theory depends only on certain chamber structures in the space of metric. In Sect. 6, we discuss some natural implications of our results on the Donaldson invariant and its transition formula, as well as some conjectures and speculations.

2. The Donaldson Polynomials and Holomorphic Yang-Mills Theory

In this section, we review the Donaldson theory, TYM theory and HYM theory on a manifold with $b_2^+(X) = 1$. Throughout this paper, for simplicity, we consider a simply connected compact algebraic surface X and the $SU(2)$ polynomial invariants only. There is no loss of generality to consider the algebraic surface only since any Kähler surface deforms to an algebraic one. We hope we can treat more general gauge group in the future.

[6] We would like to thank L. Göttsche for pointing us out, after our first version, the expanded version of [34] which contains the explicit results.

Let X be a simply connected projective algebraic surface with Kähler-Hodge metric g and associated Kähler form ω of type $(1,1)$. Since X is a projective space, we have an ample line bundle H over X whose first Chern Class $c_1(H) \in H^2(X;\mathbb{Z})$ is given by the Kähler form ω. We will occasionally confuse a line bundle with its first Chern class as well as with the the corresponding divisor. By the Kodaria projective embedding theorem, sections of some positive tensor power $H^{\otimes m}$ of the ample line bundle define projective embedding of X to some complex projective space whose hyperplane section class is Poincaré dual to $m[\omega]$. A line bundle L is ample if and only if $L \cdot L > 0$ and $L \cdot c > 0$ for every complex curve c in X, where ' $\cdot$ ' denotes the intersection pairings, i.e., $L \cdot L \equiv \int_X c_1(L) \wedge c_1(L)$. They defines a positive cone called ample cone (or Kähler cone) $Cone_X$ in the space of $H^2(X;\mathbb{R})$.

For a given metric we can decompose the space $H^2(X)$ of harmonic forms on X into the space $H^2_+(X)$ of self-dual and the space $H^2_-(X)$ of anti-self-dual harmonic forms. Similarly, we have $b_2 = b_2^+ + b_2^-$ where $b_2 = \dim H^2(X)$ and $b_2^\pm = \dim H^2_\pm(X)$. From the Hodge index theorem, we have

$$b_2^+ = 1 + 2p_g, \tag{2.1}$$

where the one-dimensional piece is spanned by the Kähler form ω and the geometric genus p_g is the number of (harmonic) holomorphic two-forms. For an arbitrary vector bundle valued two-form we have the similar decompositions, i.e. the self-dual part of the curvature two-form $F^+(A)$ can be decomposed as

$$F^+(A) = F^{2,0}(A) + f(A)\omega + F^{0,2}(A), \tag{2.2}$$

where $f(A) = \frac{1}{2}\Lambda F^{1,1}(A)$. Here Λ denotes the algebraic trace operator which is adjoint to the wedge multiplication of the Kähler form. The intersection form q_X between harmonic two-forms is an unimodular (due to the Poincaré duality) and symmetric $b_2 \times b_2$ matrix with signature $(b_2^+ - b_2^-)$. That is, upon diagonalization of q_X it has b_2^+ positive entries and b_2^- negative entries. It is also called of type (b_2^+, b_2^-). This comes from the simple fact that

$$\begin{aligned} q_X(\alpha,\alpha) = \alpha\cdot\alpha = \int_X \alpha\wedge\alpha &= \int_X \alpha^+\wedge\alpha^+ + \int_X \alpha^-\wedge\alpha^- \\ &= \int_X \alpha^+\wedge *\alpha^+ - \int_X \alpha^-\wedge *\alpha^- \\ &= \int_X |\alpha^+|^2 d\mu - \int_X |\alpha^-|^2 d\mu, \end{aligned} \tag{2.3}$$

where $*$ is the Hodge star operator and $d\mu = \omega^2/2!$ is the volume form. Thus X has $b_2^+ = 1$ if and only if $p_g = 0$ and its intersection form is of type $(1, b_2^-)$.

2.1. **The Donaldson polynomial invariants**

Let E be a complex vector bundle over X with the reduction of the structure group to $SU(2)$. The bundle E is classified by the instanton number k,

$$k =< c_2(E), X >= \frac{1}{8\pi^2}\int_X \mathrm{Tr}\,(F\wedge F) \quad \in \mathbb{Z}^+, \tag{2.4}$$

where Tr is the trace in the 2-dimensional representation and $\mathrm{Tr}\,(\xi)^2 = -|\xi|^2$ on $\mathfrak{su}(2)$. Let $\mathcal{A}_k$ be the space of all connections of E and $\mathcal{A}_k^{1,1} \in \mathcal{A}_k$ be the subspace consisting of the connections whose curvatures are of type $(1,1)$, i.e. $A \in \mathcal{A}_k^{1,1}$ iff $F^{0,2}(A) = 0$. We will call a connection A holomorphic if $A \in \mathcal{A}_k^{1,1}$. Let $\mathcal{G}$ be the group of the gauge transformations. We denote $\mathcal{M}_k(g)$ to the moduli space ASD connections with respect to a (Riemann) metric g. We denote $\mathcal{A}_k^*$ the space of irreducible connections, and thus, $\mathcal{A}_k^{*1,1} \equiv \mathcal{A}_k^* \cap \mathcal{A}_k^{1,1}$ and $\mathcal{M}_k^*(g) \equiv \mathcal{A}_k^* \cap \mathcal{M}_k(g)$. The virtual complex dimension of $\mathcal{M}_k(g)$ is $d = 4k-3$.

We briefly review a definition of the Donaldson polynomial invariants[7]. For a manifold with $b_2^+ > 0$, there are no reducible ASD connections for a generic choice of the metric. Let g be such a generic metric. Then $\mathcal{M}_k(g) = \mathcal{M}_k^*(g)$ is a smooth manifold with actual complex dimension d. Since the moduli space is rarely compact one should compactify $\mathcal{M}_k^*$ to get a fundamental homology class. The Donaldson-Uhlenbeck compactification $\overline{\mathcal{M}}_k$ of $\mathcal{M}_k^*$ is the closed subset of the embedding of $\mathcal{M}_k$ to the disjoint union[8],

$$\bigcup_{\ell=0}^{k} \mathcal{M}_\ell \times \mathrm{Sym}^{k-\ell}(X). \tag{2.5}$$

The compactified space includes the ASD moduli spaces with lower instanton numbers in the lower stratas which can contain reducible ASD connections. To get a well-defined invariants we should choose the metric such that it does not admit any reducible ASD connections with instanton numbers $1, ..., k$. The genericity of the metric always means to satisfy this additional requirement. Let

$$\mu : H_2(X;\mathbb{Z}) \longrightarrow H^2(\mathcal{M}_k^*(g);\mathbb{Z}) \tag{2.6}$$

[7] There are several other more conceptually elaborated and powerful definitions. For details, the reader can consult the excellent book [2]. The definition of the invariants in this introduction is to emphasize that one should also take care of the reducible ASD connections with lower instanton numbers.

[8] For manifold with $p_g = 0$, it was recently shown that the compactified moduli space $\overline{\mathcal{M}}_k$ can be identified with the total space itself [36].

be the Donaldson μ-map. Then we have a natural extension of μ

$$\overline{\mu} : H_2(X;\mathbb{Z}) \longrightarrow H^2(\overline{\mathcal{M}}_k(g);\mathbb{Z}). \tag{2.7}$$

For $k > 1$ (the stable range), the compactified moduli space carries the fundamental homology class. The Donaldson polynomial $\overline{q}_{X,g,k}$,

$$\overline{q}_{X,g,k}(\Sigma_1, ..., \Sigma_{4k-3}) =< \overline{\mu}(\Sigma_1) \smile ... \smile \overline{\mu}(\Sigma_{4k-3}), [\overline{\mathcal{M}}_k(g)] > \tag{2.8}$$

defines the map;

$$H_2(X,\mathbb{Z}) \times \cdots \times H_2(X,\mathbb{Z}) \to \mathbb{Z},$$

which is well-defined for the generic metric g.

The above definition of the Donaldson polynomial is basically the same as those for manifolds with $b_2^+ > 1$. The special feature of the manifold with $b_2^+ = 1$ is that the polynomial actually depends on the metric. To show that the polynomial (2.8) is metric independent, one should consider a smooth generic path g_t of metrics joining the two generic metrics and show that its value does not change according to the variation of the metric. This amounts to show that the one parameter family of the moduli space $\mathcal{M}_k(g_t)$ or rather its compactification $\overline{\mathcal{M}}_k(g_t)$ does not depend on g_t at the level of homology. This can be ensured if there is no point in g_t which admits the reducible ASD connections.

A $SU(2)$ connection is reducible if the $SU(2)$-bundle reduces to direct sum of line bundles $E = \zeta \oplus \zeta^{-1}$. Since the bundle E is classified by the second Chern-Class, from the Whitney formula, we have the following condition for the bundle reduction

$$k =< c_2(E), X >= -c_1(\zeta) \cdot c_1(\zeta^{-1}) = -c_1(\zeta) \cdot c_1(\zeta) = -\zeta \cdot \zeta \tag{2.9}$$

where $c_1(\zeta) = H^2(X,\mathbb{Z})$. Since k should be positive definite to admit ASD connection, the self-intersection of line bundle should be negative definite to solve above equation. Let ω_g be the family of the self-dual two-forms associated with a metric g. The metric admits reducible ASD connection if and only if

$$\int_X c_1(\zeta) \wedge \omega_g = 0. \tag{2.10}$$

If the number of self-dual harmonic form, whose self-intersection is positive definite, is greater than 0, we can avoid the reducible connection by perturbing metric such that there

are no reducible connection for generic choice of metric. Now we consider a smooth generic path g_t of the metric joining two generic metrics. For manifold with $b_2^+ > 1$ such a path can always avoid metric admitting the reducible ASD connection. However, for manifold with $b_2^+ = 1$ there can be at least finite number of the points in g_t which admit reducible ASD connections since the subspace of ASD two-forms in the space $H^2(X;\mathbb{R})$ has the codimension one.

Since we are considering the compactified moduli space, we should also consider all the bundle reductions given by

$$-k \leq \zeta \cdot \zeta \leq -1, \tag{2.11}$$

where $\zeta \in H^2(X;\mathbb{Z})$. Then, the compactified moduli space $\overline{\mathcal{M}}_k(g)$ does not contain any reducible ASD connection if and only if $\int_X c_1(\zeta) \wedge \omega_g \neq 0$ for all ζ satisfying (2.11). A proper definition of the Donaldson polynomials requires the systematic understanding of the appearance of the reducible ASD connection as one varies the metric.

2.2. **The chamber structure**

Let Ω_X be the positive cone in $H^2(X;\mathbb{R})$ defined by

$$\Omega_X = \{\theta \in H^2(X;\mathbb{R}) | \theta \cdot \theta > 0\}. \tag{2.12}$$

Since the intersection form q_X is of type $(1,n)$ the positive cone has two connected components. For each element ζ satisfying (2.11) one defines the wall $W_\zeta = W_{-\zeta}$ by the intersection of the hyperplane $\zeta^\perp \in H^2(X;\mathbb{R})$ orthogonal to ζ, i.e., $\zeta \cdot \zeta^\perp = 0$, with Ω_X. We denote $\mathcal{W}_\ell$ by the collection of walls defined by all $\zeta \in H^2(X;\mathbb{Z})$ satisfying $\zeta \cdot \zeta = -\ell$. We also denote the system $\overline{\mathcal{W}}_k$ of walls by

$$\overline{\mathcal{W}}_k = \bigcup_{1 \leq \ell \leq k} \mathcal{W}_\ell. \tag{2.13}$$

The set $\overline{\mathcal{C}}_X^k$ of chambers[9] is the set of the connected components of Ω_X after removing $\overline{\mathcal{W}}_k$.

[9] The physicist reader may find it easy to understand the chamber structure by an analogy with $(1+n)$ dimensional Minkowski space with metric $diag(1,-1,\ldots,-1)$. This analogy is rigorous if the intersection form is odd. The vector space $H^2(X;\mathbb{R})$ corresponds to the Minkowski space. The intersection form is just the metric form and the positive cone corresponds to the future light-cone. We consider the future cone which contains the ample cone. An integral class $\zeta \in H^2(X;\mathbb{Z})$ corresponds to a vector and its intersection number to the norm squared of the vector in the Minkowski space. An integral class with negative self-intersection number corresponds to a space-like vector. Then the wall is defined by a space-like hyperplane. Definitely, the hyperplane orthogonal to a vector intersect with the time-like space if and only if the vector is space-like.

Without loss of generality one can only consider one of the two connected components of Ω_X which contains the Kähler cone. It is convenient to choose a level set $H(q) \in \Omega_X$ defined by the n-dimensional hyperbolic space satisfying $\theta \cdot \theta = 1$. A metric g determines a line[10] in Ω_X made up of the cohomology classes represented by g-self-dual harmonic two forms ω_g. Let $[\omega_g]$ be the point of the line intersects with $H(q)$. Now we can see that the compactified moduli space contains reducible ASD connection if and only if $[\omega_g]$ lies in one of the walls.

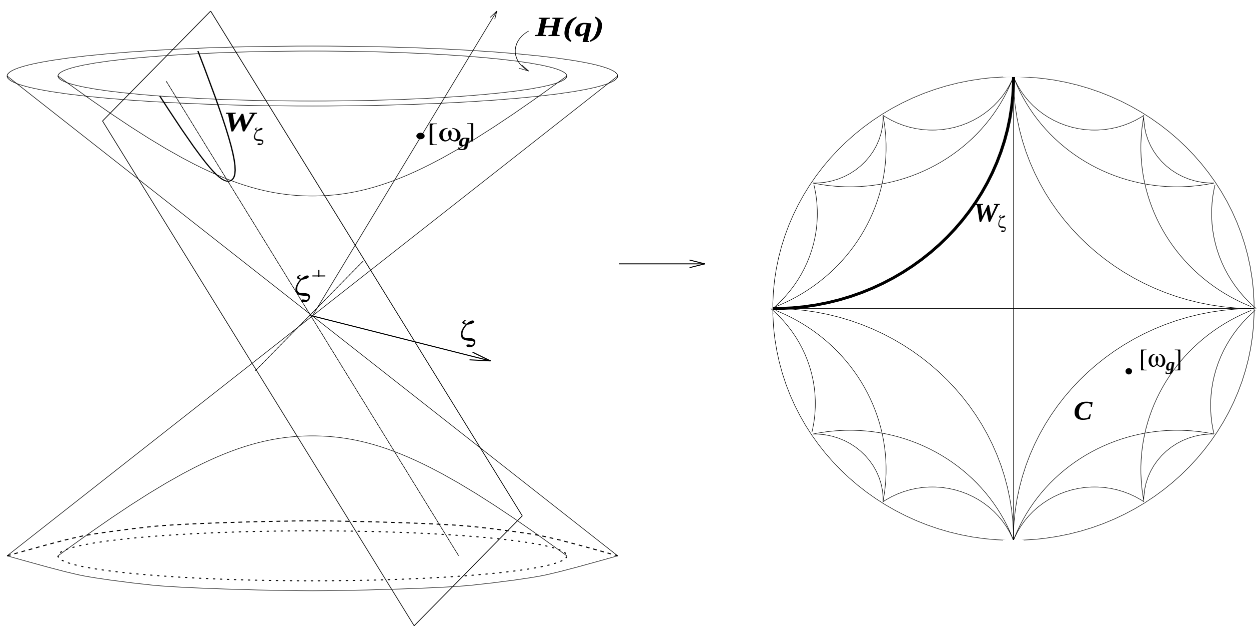

Fig. 1. A typical chamber structure [3] for a manifold X of type $(1,2)$ and $k=1$ i.e., $\zeta\cdot\zeta=-1$. The right-hand side is the the Poincaré model for $H(q)$ and C denote the chamber containing ω_g. The pattern repeats to infinity.

For a smooth path g_t of metrics we have the corresponding path $[\omega_{g_t}]$ in $H(q)$. Although the moduli space $\mathcal{M}_k(g_t)$ and its compactification certainly depends on t, it may not be changed at the level of homology. If $[\omega_{g_t}]$ is contained in a chamber, the homology class of $[\overline{\mathcal{M}}_k(g_t)]$ does not depend on g_t. On the other hand, if ω_{g_t} crosses the walls, special things happen such that the moduli space is changed even at the level of homology. The variation of the homology class of the moduli space is essentially due to the appearance of the reducible ASD connection.

[10] Note that the space of self-dual harmonic two-forms is one-dimensional.

After picking a generic metric g such that $[\omega_g]$ lies in one of the chambers $C \in \overline{\mathcal{C}}^k_X$ and consider

$$\overline{q}_{X,g,k}(\Sigma^{4k-3}) \tag{2.14}$$

an element of $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$, we can define the map

$$\overline{\Gamma}^k_X : C \rightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z})), \tag{2.15}$$

which depends only on the chamber structure $\overline{\mathcal{C}}^k_X$ in Ω_X. The $SU(2)$-invariants of X introduced by Donaldson [3] and extended by Mong [6] and by Kotschick-Morgan [5][7] are the assignments

$$\overline{\Gamma}^k_X : \overline{\mathcal{C}}^k_X \rightarrow \mathrm{Sym}^{4k-3}(H^2(X,\mathbb{Z})), \tag{2.16}$$

The polynomial $\overline{\Gamma}^k_X(C)$ depends only on the chamber with following properties:

i) $\overline{\Gamma}^k_X(-C) = -\overline{\Gamma}^k_X(C)$

ii) If $f : X_1 \rightarrow X_2$ is an orientation preserving diffeomorphism between two such manifolds, then $\overline{\Gamma}^k_{X_1}(f^*(C)) = f^*(\overline{\Gamma}^k_{X_2}(C))$.

The computation of the Donaldson invariant amounts to determine the invariant in a certain chamber and to find a general transition formula its variations when $[\omega_{g_t}]$ cross the walls. The Donaldson polynomial invariants $\overline{q}_{X,g,k}(\Sigma^{4k-3})$ may not be well-defined if $[\omega_g]$ lies one of the walls due to the singularity in the moduli space. One can also extend the definition of the invariants including the four-dimensional class such that $\overline{q}_{X,g,k}(\Sigma^{4k-3-2r}(pt)^r)$. We will denote $\overline{\Gamma}^{k,r}_X(C)$ for the corresponding assignments which also depend only on the chamber structure. This extension becomes more problematic when the moduli space has singularity.

2.3. **The topological Yang-Mills theory**

To begin with, we recall the $N = 2$ TYM theory on compact Kähler surfaces [18][22][20]. The theory has $N = 2$ global supersymmetry whose conserved charges $\mathbf{s}$ and $\bar{\mathbf{s}}$ can be identified with the operators of $\mathcal{G}$-equivariant Dolbeault cohomology of $\mathcal{A}$. The algebra for the basic multiplet $(A', A'', \psi, \bar{\psi}, \varphi)$ [18] is

$$\begin{aligned}
&\mathbf{s}A' = -\psi, && \mathbf{s}\psi = 0, && \\
&\bar{\mathbf{s}}A' = 0, && \bar{\mathbf{s}}\psi = -i\partial_A\varphi, && \bar{\mathbf{s}}\varphi = 0, \\
&\mathbf{s}A'' = 0, && \mathbf{s}\bar{\psi} = -i\bar{\partial}_A\varphi, && \mathbf{s}\varphi = 0, \\
&\bar{\mathbf{s}}A'' = -\bar{\psi}, && \bar{\mathbf{s}}\bar{\psi} = 0, &&
\end{aligned} \tag{2.17}$$

where A' and A'' denote the holomorphic and anti-holomorphic parts of the connection one-form $A = A' + A''$, $\psi \in \Omega^{1,0}(\mathfrak{g}_E)$, $\bar{\psi} \in \Omega^{0,1}(\mathfrak{g}_E)$ and $\varphi \in \Omega^0(\mathfrak{g}_E)$. Note that ψ can be identified with holomorphic (co)tangent vectors on $\mathcal{A}$. The fields have additional quantum numbers characterized by the degree $(*,*)$. The operator $\mathbf{s}$ carries the degree $(1,0)$ and $\bar{\mathbf{s}}$ carries the degree $(0,1)$. Assigning the degree $(0,0)$ to the connection A, φ is of degree $(1,1)$. In terms of the equivariant cohomology the above algebra can be represented as follows. We let $\Omega^{*,*}(\mathcal{A})$ be the Dolbeault complex on $\mathcal{A}$. Now we interpret $\mathrm{Fun}(Lie(\mathcal{G}))$ to the algebra of polynomial functions generated by φ^a. Then the desired Dolbeault model of the $\mathcal{G}$-equivariant complex is $\Omega^{*,*}_{\mathcal{G}} = (\Omega^{*,*}(\mathcal{A}) \otimes \mathrm{Fun}(\mathcal{G}))^{\mathcal{G}}$. The associated differential operators with the degrees $(1,0)$ and $(0,1)$ are $\mathbf{s}$ and $\bar{\mathbf{s}}$, represented by

$$
\begin{aligned}
\mathbf{s} &= -\sum_i \psi^i \frac{\partial}{\partial A'^i} + i \sum_{\bar{i},a} \varphi^a V_a^{\bar{i}} \frac{\partial}{\partial \overline{\psi}^{\bar{i}}}, \\
\bar{\mathbf{s}} &= -\sum_{\bar{i}} \bar{\psi}^{\bar{i}} \frac{\partial}{\partial A''^{\bar{i}}} + i \sum_{i,a} \varphi^a V_a^i \frac{\partial}{\partial \psi^i},
\end{aligned}
\tag{2.18}
$$

where $i, \bar{i}$ are the local holomorphic and anti-holomorphic indices tangent to $\mathcal{A}$, respectively. We have

$$
\mathbf{s}^2 = 0, \qquad \mathbf{s}\bar{\mathbf{s}} + \bar{\mathbf{s}}\mathbf{s} = -i\varphi^a \mathcal{L}_a, \qquad \bar{\mathbf{s}}^2 = 0,
\tag{2.19}
$$

Thus, $\{\mathbf{s}, \bar{\mathbf{s}}\} = 0$ on the $\mathcal{G}$-invariant subspace $\Omega^{*,*}_{\mathcal{G}}$ of $\Omega^{*,*}(\mathcal{A}) \otimes \mathrm{Fun}(Lie(\mathcal{G}))$. *We define the $\mathcal{G}$-equivariant Dolbeault cohomology $H^{*,*}_{\mathcal{G}}(\mathcal{A})$ by the pairs $(\Omega^{*,*}_{\mathcal{G}}(\mathcal{A}),\ \bar{\mathbf{s}})$.* It was shown that for manifold with $p_g = 0$ the $\mathbf{s}$-cohomology is isomorphic to the $\bar{\mathbf{s}}$ cohomology [20].

The action functional of $N = 2$ TYM theory can be viewed as the Dolbeault equivariant cohomological version of the Mathai-Quillen representative of the universal Thom class[11] of the infinite dimensional bundle $\mathcal{A} \to \mathcal{A}/\mathcal{G}$. We consider the vector space V of $\mathfrak{g}_E$-valued self-dual two-forms with a linear $\mathcal{G}$ action on it. Then we can form a homology quotient $E_{\mathcal{G}} = \mathcal{A} \times_{\mathcal{G}} V$. Then there exists a equivariant map

$$
F^+ : \mathcal{A} \longrightarrow V \quad \text{by} \quad A \longrightarrow F^+(A),
\tag{2.20}
$$

[11] The standard reference on the de Rham equivariant cohomology is the book [37]. The relation with the topological field theory was studied in [38][39][40]. Recently, an extensive and consistent review on the cohomological field theory based on the equivariant de Rham cohomology appeared [41]. The paper [23] contains a self-contained introduction on the subject as well as a generalization.

where $F^+(A)$ denotes the self-dual part of the curvature two form $F(A)$ and defines a section s of $E_{\mathcal{G}}$. Now, the moduli space of ASD connections is the zero set of the section s. In the Kähler geometry, we can decompose the vector space V into the vector spaces $V = V^{2,0} \oplus V^{1,1}_{\omega} \oplus V^{0,2}$ of $(2,0)$-forms, $(1,1)$-forms proportional to the Kähler form and $(0,2)$-forms. Using the natural complex structure on $\mathcal{A}$ induced from X, we can also decompose the section s (the equivariant map) into

$$
\begin{aligned}
&F^{2,0} : \mathcal{A}' \longrightarrow V^{2,0} \quad \text{by} \quad A' \longrightarrow F^{2,0}(A'),\\
&F^{1,1}_{\omega} : \mathcal{A} \longrightarrow V^{1,1}_{\omega} \quad \text{by} \quad A \longrightarrow f(A', A'')\omega,\\
&F^{0,2} : \mathcal{A}'' \longrightarrow V^{0,2} \quad \text{by} \quad A'' \longrightarrow F^{0,2}(A''),
\end{aligned}
\tag{2.21}
$$

where $f(A) = \frac{1}{2}\Lambda F^{1,1}(A)$.

To write the Mathai-Quillen representative of the Universal Thom class, we should introduce the set of anti-ghost multiplets for each components of the above map. Geometrically, the anti-ghost multiplets are various equivariant differential forms living in the dual vector space of V and their spectrum and algebra can be uniquely determined in terms of the Dolbeault model of the equivariant cohomology [21](see also [18][20]). This leads to a commuting anti-ghost $\bar{\varphi} \in \Omega^0(\mathfrak{g}_E)$ living in the dual vector space $V^{*1,1}_{\omega}$ with degree $(-1,-1)$. Then we have multiplet $(\bar{\varphi}, i\chi^0, -i\bar{\chi}^0, H^0)$ with transformation laws

$$
\begin{aligned}
&\mathbf{s}\bar{\varphi} = -i\chi^0, &&\mathbf{s}\chi^0 = 0,\\
&\bar{\mathbf{s}}\bar{\varphi} = i\bar{\chi}^0, &&\bar{\mathbf{s}}\bar{\chi}^0 = 0,\\
&\mathbf{s}\bar{\chi}^0 = H^0 - \tfrac{1}{2}[\varphi, \bar{\varphi}], &&\mathbf{s}H^0 = -\tfrac{i}{2}[\varphi, \chi^0],\\
&\bar{\mathbf{s}}\chi^0 = H^0 + \tfrac{1}{2}[\varphi, \bar{\varphi}], &&\bar{\mathbf{s}}H^0 = -\tfrac{i}{2}[\varphi, \bar{\chi}^0].
\end{aligned}
\tag{2.22}
$$

We also have an anti-commuting anti-ghost $\chi^{2,0}$ in the dual vector space $V^{2,0}$ with degree $(-1,0)$ and an anti-commuting anti-ghost $\bar{\chi}^{0,2}$ living in the dual vector space $V^{*0,2}$ with degree $(0,-1)$ with transformation laws

$$
\begin{aligned}
&\mathbf{s}\chi^{2,0} = 0, &&\mathbf{s}H^{2,0} = -i[\varphi, \chi^{2,0}],\\
&\bar{\mathbf{s}}\chi^{2,0} = H^{2,0}, &&\bar{\mathbf{s}}H^{2,0} = 0,\\
&\mathbf{s}\bar{\chi}^{0,2} = H^{0,2}, &&\mathbf{s}H^{0,2} = 0,\\
&\bar{\mathbf{s}}\bar{\chi}^{0,2} = 0, &&\bar{\mathbf{s}}H^{0,2} = -i[\varphi, \bar{\chi}^{0,2}].
\end{aligned}
\tag{2.23}
$$

The action functional (the universal Thom class) is given by

$$S = - i\mathbf{s}\left(\frac{1}{h^2}\int \mathrm{Tr}\,\bar{\chi}^{0,2}\wedge *F^{2,0}\right) - i\bar{\mathbf{s}}\left(\frac{1}{h^2}\int \mathrm{Tr}\,\chi^{2,0}\wedge *F^{0,2}\right) - (\mathbf{s}\bar{\mathbf{s}} - \bar{\mathbf{s}}\mathbf{s})\left(\frac{1}{h^2}\int \mathrm{Tr}\,\chi^{2,0}\wedge *\bar{\chi}^{0,2}\right)$$
$$- \frac{(\mathbf{s}\bar{\mathbf{s}} - \bar{\mathbf{s}}\mathbf{s})}{2}\left(\frac{1}{h^2}\int \mathrm{Tr}\left(\bar{\varphi}f + \chi^0\bar{\chi}^0\right)\omega^2\right). \tag{2.24}$$

A small calculation gives

$$S = \frac{1}{h^2}\int_X \mathrm{Tr}\left[-\frac{1}{2}F^{2,0}\wedge *F^{0,2} + i\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + i\bar{\chi}^{0,2}\wedge *\partial_A\psi - 2i[\varphi,\chi^{2,0}]\wedge *\bar{\chi}^{0,2}\right.$$
$$\left. - \left(\tfrac{1}{2}f^2 - 2i[\varphi,\chi^0]\bar{\chi}^0 - \bar{\chi}^0\partial_A^*\psi + \chi^0\bar{\partial}_A^*\bar{\psi} - \frac{1}{2}[\varphi,\bar{\varphi}]^2 + \frac{1}{2}\bar{\varphi}\left(d_A^*d_A\varphi - 2i\Lambda[\psi,\bar{\psi}]\right)\right)\frac{\omega^2}{2!}\right]. \tag{2.25}$$

where we have integrated out auxiliary fields $H^{2,0}$, H^0, $H^{0,2}$ and used the Kähler identities,

$$\bar{\partial}_A^* = i[\partial_A,\Lambda], \qquad \partial_A^* = -i[\bar{\partial}_A,\Lambda], \tag{2.26}$$

The bosonic kinetic terms of the action is

$$S = -\frac{1}{h^2}\int \mathrm{Tr}\left(\frac{1}{2}|F^+|^2 + \frac{1}{2}|d_A\varphi|^2\right)d\mu + \ldots. \tag{2.27}$$

The first term is the norm square of the equivariant section of $E_{\mathcal{G}}$. In the $h^2 \to 0$, the dominant contribution of the path integral comes from the instantons $F^+(A) = 0$ and the configuration satisfying

$$d_A\varphi = 0. \tag{2.28}$$

If there is a non-zero solution φ of the equation (2.28) , it means the instanton A is reducible (abelian). Then the $SU(2)$ group reduces to the $U(1)$ subgroup,

$$\varphi = \varphi_c T^3 = -\frac{i}{2}\begin{pmatrix}\varphi_c & 0\\ 0 & -\varphi_c\end{pmatrix} \in \mathfrak{su}(2). \tag{2.29}$$

Note also that, for reducible connection, the superpotential term

$$+\frac{1}{h^2}\int \mathrm{Tr}\left(\frac{1}{2}[\varphi,\bar{\varphi}]^2\right)d\mu, \tag{2.30}$$

vanishes, which corresponds to the flat direction in the physical terms. If the reducible instantons appear, the path integral has additional contribution from the vector space of the φ-zero-modes. Of course the moduli space of ASD connections becomes singular.

Then, the semi-classical description breaks down due to the singularity and the topological interpretation of the theory can be invalidated due to the non-compactness of the vector space of the φ-zero-modes[24]. One can also view the localization of the path integral by the fixed point locus;

$$
\begin{array}{ll}
\begin{cases} \mathbf{s}\psi = -i\partial_A\varphi = 0 \\ \bar{\mathbf{s}}\bar{\psi} = -i\bar{\partial}_A\varphi = 0 \end{cases} & \Longrightarrow \quad d_A\varphi = 0 \\
\begin{cases} \mathbf{s}\bar{\chi}^{0,2} = -\frac{i}{2}F^{2,0}(A) = 0 \\ \bar{\mathbf{s}}\chi^{2,0} = -\frac{i}{2}F^{0,2}(A) = 0 \\ \mathbf{s}\bar{\chi}^{0} = -\frac{i}{2}f(A) - \frac{1}{2}[\varphi,\bar{\varphi}] = 0 \\ \bar{\mathbf{s}}\chi^{0} = -\frac{i}{2}f(A) + \frac{1}{2}[\varphi,\bar{\varphi}] = 0 \end{cases} & \Longrightarrow \quad \begin{cases} F_A^+ = 0 \\ [\varphi,\bar{\varphi}]^2 = 0 \end{cases}
\end{array}
\tag{2.31}
$$

Picking a two-dimensional class $\Sigma \in H_2(X;\mathbb{Z})$, one can define a topological observable

$$
\mu(\Sigma) \equiv \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\left(i\varphi F + \psi\wedge\bar{\psi}\right) \equiv \frac{1}{4\pi^2}\int_X \mathrm{Tr}\left(i\varphi F + \psi\wedge\bar{\psi}\right)\wedge\alpha_\Sigma, \tag{2.32}
$$

where $\alpha_\Sigma \in H^2(X;\mathbb{Z})$ is Poincaré dual to Σ. This observable is the field theoretic representation of the Donaldson's μ-map (2.6). One also defines the observable Θ corresponding to the four-dimensional class, $\mu(pt)$,

$$
\Theta = \frac{1}{8\pi^2}\int_X \frac{\omega^2}{2!}\mathrm{Tr}\,\phi^2. \tag{2.33}
$$

Assuming that there is no reducible instanton, the correlation function

$$
\langle\mu(\Sigma_1)\cdots\mu(S_s)\Theta^r\rangle = \frac{1}{\mathrm{vol}(\mathcal{G})}\int \mathcal{D}X\, e^{-S}\,\mu(\Sigma_1)\cdots\mu(\Sigma_s)\Theta^r, \tag{2.34}
$$

is the path integral representation of the Donaldson invariant $\overline{q}_{k,X}(\Sigma_1,\ldots,\Sigma_s,(pt)^r)$. Due to the ghost number anomaly, the correlation function (2.34) always vanish unless $\dim_{\mathbb{C}}\mathcal{M}_k = s + 2r$.

It is not entirely clear how the path integral (2.34) takes care of the non-compactness of the instanton moduli space. However, at least for manifolds with $b_2^+ \geq 3$, Witten's explicit results show that the path integral correctly leads to the Donaldson invariants.

Clearly the path integral localizes to the moduli space of ASD connections rather than the compactified one. We do not know any clear reasoning why the path integral correctly reproduce the Donaldson invariant. One may argue that the additional space added for the compactification does not contribute to the Donaldson invariants. In fact more elaborated definitions of the invariants, compared to that of (2.8), clearly indicate such a property under certain conditions [1][2]. For a manifold with $b_2^+ = 1$, on the other hand, the path integral does not exactly represent the Donaldson invariants. This can be easily seen if one considers the reducible connections. Clearly the path integral localizes to the moduli space of ASD connections rather than the compactified one. In the path integral approach, then, we only need to worry about the reducible connections corresponding to the bundle reduction (2.9). In any case, one can insist that the path integral approach is well-defined whatever properties the instanton moduli space has. We believe that Witten's explicit results on the $b_2^+ \geq 3$ cases and our partial result in this paper for $b_2^+ = 1$ support such a viewpoint.

We would like to add the following remarks.

i) The form (2.24) of the action functional has been uniquely determined. We could correctly recover every term in the action functional of the $N = 2$ super-Yang-Mills theory. The global supersymmetry transformation laws and the action functional are rigorously identical to those of the twisted theory [22]. On the other hand, The usual approach based on the de Rham model of the equivariant cohomology or the $N = 1$ global supersymmetry does not leads to the complete determination of the anti-ghost multiplets and the correct action functional. One should add, so called, projection term and non-minimal term [41].

ii) The usual approach to TYM theory on Kähler surface based on the $N = 1$ global supersymmetry can not explain the perturbation of Witten, breaking the $N = 2$ symmetry down to $N = 1$ symmetry [22][20]. Unfortunately, the perturbation is not applicable for manifolds with $p_g = 0$.

2.4. The holomorphic Yang-Mills theory

An obvious way out of the difficulty of the TYM theory with the reducible connections is to eliminate the zero-mode of φ from the theory as originally suggested by Witten in the two-dimensional model of the TYM theory [26]. The remarkable fact is that his method eventually leads to a non-abelian localization theorem of the theory of the equivariant (de

Rham) cohomology[12]. The HYM theory is an analogous prescription for the Donaldson theory and it is related to a Dolbeault equivariant cohomological version of the non-Abelian localization theorem [18][20]. Consequently the HYM theory is a suitable model for the Donaldson invariants on Kähler surface with $b_2^+ = 1$.

The basic observation is that the reduction of the path integral of the TYM theory to the instanton moduli space is achieved by the following two steps; i) restriction of $\mathcal{A}$ to $\mathcal{A}^{1,1}$, ii) restriction of $\mathcal{A}^{1,1}$ to the solution space of ASD connections and reduction to the instanton moduli space by dividing by the gauge group $\mathcal{G}$. Then the second step can be replaced with the symplectic reduction. Now we we can deform the $(1,1)$ part of the action by the one-parameter family of the action,

$$S(t) = S + t(\mathbf{s}\bar{\mathbf{s}} - \bar{\mathbf{s}}\mathbf{s})\left(-\frac{1}{h^2}\int_M \frac{\omega^2}{2!}\operatorname{Tr}\bar{\varphi}^2\right), \tag{2.35}$$

where t is a real positive deformation parameter. After some Gaussian integrals we are left with[13]

$$\begin{aligned} S(t) =& -i\mathbf{s}\left(\frac{1}{h^2}\int \operatorname{Tr}\bar{\chi}^{0,2}\wedge * F^{2,0}\right) - i\bar{\mathbf{s}}\left(\frac{1}{h^2}\int \operatorname{Tr}\chi^{2,0}\wedge * F^{0,2}\right) \\ &+ \frac{\mathbf{s}\bar{\mathbf{s}} - \mathbf{s}\bar{\mathbf{s}}}{2}\left(\frac{1}{2h^2 t}\int_M \frac{\omega^2}{2!}\operatorname{Tr} f^2\right). \end{aligned} \tag{2.36}$$

The action functional of the HYM theory is defined by

$$\begin{aligned} S(t)_H =& \frac{1}{h^2}\int_X \operatorname{Tr}\left[-iH^{2,0}\wedge *F^{0,2} - iH^{0,2}\wedge *F^{2,0} + i\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + i\bar{\chi}^{0,2}\wedge *\partial_A\psi\right] \\ &- \frac{1}{4\pi^2}\int_X \operatorname{Tr}\left(i\varphi F + \psi\wedge\bar{\psi}\right)\wedge\omega - \frac{\varepsilon}{8\pi^2}\int_X \frac{\omega^2}{2!}\operatorname{Tr}\varphi^2 \\ &+ \frac{1}{4\pi^2\varepsilon}\int_X \operatorname{Tr}\left(F^{2,0}\wedge F^{0,2} + \frac{1}{2}F^{1,1}\wedge F^{1,1}\right) \\ &+ \frac{\mathbf{s}\bar{\mathbf{s}} - \mathbf{s}\bar{\mathbf{s}}}{2}\left(\frac{1}{2h^2 t}\int_M \frac{\omega^2}{2!}\operatorname{Tr} f^2\right), \end{aligned} \tag{2.37}$$

where ε is positive number.

Several remarks are in order.

---

[12] The equivariant localization was studied in the mathematical literatures [42][43][44][45][46] for the abelian version and [47][48][49] for the non-abelian version. The physics oriented reader may find the review [41] the most readable. See also [50]

[13] Here, we choose the delta-function gauge for simplicity.

i) The first line of the action is identical to the part of the TYM action (Thom class) which leads to a clear cut reduction of $T^*\mathcal{A}$ to $T^*\mathcal{A}^{1,1}$ without any quantum correction. Thus we can regard the theory as the one defined on $T^*\mathcal{A}^{1,1}$. Similar hybrid model of the de Rahm model was suggested in [51] independently to [25].

ii) The term

$$\tilde{\omega} \equiv \frac{1}{4\pi^2}\int_X \mathrm{Tr}\left(i\varphi F + \psi\wedge\bar{\psi}\right)\wedge\omega \tag{2.38}$$

is the equivariant extension of the Kähler form $\frac{1}{4\pi^2}\int_X \mathrm{Tr}\,(\psi\wedge\bar{\psi})\wedge\omega$ on $\mathcal{A}^{1,1}$. It also define a two dimensional class $\mu(H)$ of the Donaldson invariants associated to the ample class H.

iii) The term

$$\Theta \equiv \frac{1}{8\pi^2}\int_X \frac{\omega^2}{2!}\,\mathrm{Tr}\,\varphi^2 \tag{2.39}$$

is the four dimensional class of the Donaldson invariant.

iv) The term $\frac{1}{2h^2 t}\int_M \frac{\omega^2}{2!}\,\mathrm{Tr}\, f^2$ is proportional to the norm squared $<,>$ of the moment map $\mathfrak{m} : \mathcal{A}^{1,1} \to \Omega^0(\mathfrak{g}_E)^*$,

$$\mathfrak{m}(A) = -\frac{1}{4\pi^2}F_A^{1,1}\wedge\omega = -\frac{1}{4\pi^2}f\omega^2, \tag{2.40}$$

where $\Omega^0(\mathfrak{g}_E)^* = \Omega^4(\mathfrak{g}_E)$ denotes dual of $\Omega^0(\mathfrak{g}_E) = Lie(\mathcal{G})$. Since the path integral is independent of t, we can set $t \to 0$, and hence the path integral gets contributions only from the critical set of the function $<\mathfrak{m},\mathfrak{m}>$, i.e., $< f, d_A f >= 0$.

We set $t \to \infty$ so that we can omit the equivariantly exact form. The resulting action,

$$S_H = -\frac{1}{4\pi^2}\int_X \mathrm{Tr}\left(i\varphi F^{1,1} + \psi\wedge\bar{\psi}\right)\wedge\omega - \frac{\varepsilon}{8\pi^2}\int_X \frac{\omega^2}{2!}\,\mathrm{Tr}\,\varphi^2 + \frac{k}{\varepsilon} + \cdots, \tag{2.41}$$

after the Gaussian integral over φ, is identical to the physical Yang-Mills theory restricted to the space $\mathcal{A}^{1,1}$ of holomorphic connection. It is proportional to the normed-square of moment map up to topological terms. The real number ε corresponds to the coupling constant. The classical equation of motion is given by

$$F^{2,0}(A) = F^{0,2}(A) = 0, \qquad d_A f(A) = 0, \tag{2.42}$$

which is the Yang-Mills equation of motion on $\mathcal{A}^{1,1}$. The partition function of the HYM theory exactly reduces to an infinite dimensional non-abelian equivariant integration formula,

$$Z(\varepsilon,k) = e^{-\frac{k}{\varepsilon}} \times \frac{1}{\mathrm{vol}(\mathcal{G})}\int_{T^*\mathcal{A}^{1,1}} \mathcal{D}A'\,\mathcal{D}A''\,\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{D}\varphi \\ \times \exp\biggl(\frac{1}{4\pi^2}\int_M \mathrm{Tr}\left(i\varphi F^{1,1} + \psi\wedge\bar{\psi}\right)\wedge\omega + \frac{\varepsilon}{8\pi^2}\int_M \frac{\omega^2}{2!}\mathrm{Tr}\,\varphi^2\biggr). \tag{2.43}$$

In the limit $\varepsilon \to 0$, the partition function is related to certain expectation value of the original TYM theory, up to the exponentially small terms,

$$\begin{aligned} Z(\varepsilon,k) &= e^{-\frac{k}{\varepsilon}}\Big\langle \exp(\tilde{\omega}+\varepsilon\Theta)\Big\rangle + \text{exponentially small terms} \\ &= e^{-\frac{k}{\varepsilon}}\sum_{r=0}^{[(4k-3)/2]}\frac{\varepsilon^r}{(d-2r)!r!}\Big\langle \tilde{\omega}^{d-2r}\Theta^r\Big\rangle + \text{exponentially small terms}, \end{aligned} \tag{2.44}$$

We assumed that the path integrals are defined with respect to the metric whose Kähler form lies in one of the chambers. Otherwise, as we will show in Sect. 4, the partition function contains a non-analytic term proportional to $\varepsilon^{2k-3/2}$, which is the contribution due to the reducible instantons.

There is another way of justifying that the path integral can be expressed as the sum of contributions of the critical points. The action functional has the global $N = 2$ supersymmetry, thus, we can use the fixed point theorem of Witten. The important fixed point equation is

$$\bar{\mathbf{s}}\varphi = -i\partial_A\varphi = 0, \qquad \mathbf{s}\varphi = -i\bar{\partial}_A\varphi = 0. \tag{2.45}$$

This equation shows that non-zero solutions for φ (the zero-modes of φ) appear for reducible connections. By eliminating φ using Gaussian integral gives

$$2if + \varepsilon\varphi = 0. \tag{2.46}$$

Combining the above two equations we are led to the fixed point equation $d_A f = 0$. *Thus, the zero-modes of* φ *are no longer associated with the reducible instanton, rather they are mapped into higher critical points.*

We can further apply the fixed point theorem to calculate the partition function. The path integral can be done by evaluating exactly at the fixed point locus and by evaluating one-loop contribution of the normal modes to the fixed point locus[27]. This is the

basic method of our calculation. The partition function of the two-dimensional physical Yang-Mills theory, which is the similar low dimensional cousin of HYM theory, has been calculated also by adopting the fixed point theorem [52][53].

Before moving to the next section, we should add a *cautionary remark* on the dual roles of $\tilde{\omega}$ (eq.(2.38)). We are interested in the variation of $<\mu(\Sigma)^{d-2r}(pt)^r>$ where Σ is an arbitrary fixed element of $H_2(X;\mathbb{Z})$ according to the changes of metric. On the other hand, the Kähler form ω and its Poincaré dual H in the action and in $\tilde{\omega}$ vary as we change the metric. This amounts to using the different two dimensional classes rather than fixed one. Thus, we should read the relation (2.44) very carefully. We will return this in Sect. 5.

3. The Path Integral

We use the action functional of the HYM theory in the delta function gauge

$$\begin{aligned} S_H =& \frac{1}{h^2}\int_X \operatorname{Tr}\left[-iH^{2,0}\wedge *F^{0,2} - iH^{0,2}\wedge *F^{2,0} + i\chi^{2,0}\wedge *\bar{\partial}_A\bar{\psi} + i\bar{\chi}^{0,2}\wedge *\partial_A\psi\right] \\ &- \frac{1}{4\pi^2}\int_X \operatorname{Tr}\left(i\varphi F^{1,1} + \psi\wedge\bar{\psi}\right)\wedge\omega - \frac{\varepsilon}{8\pi^2}\int_X \frac{\omega^2}{2!}\operatorname{Tr}\varphi^2 \end{aligned} \tag{3.1}$$

The partition function of the HYM theory has contributions from the two branches.

A. The non-abelian branch : $\varphi_f = 0$ where the full $SU(2)$ symmetry is restored, i.e., the irreducible ASD connections.

B. The abelian branch : $\varphi_f \neq 0$ where the non-abelian symmetry breaks down to abelian one, i.e., reducible holomorphic connections including reducible ASD connections, if any.

Then, we can divide the partition function $Z(\varepsilon,k)$ of $N=2$ HYM theory as the sum of contributions of the two branches,

$$Z(\varepsilon,k) = Z_{\mathbf{A}}(\varepsilon,k) + Z_{\mathbf{B}}(\varepsilon,k). \tag{3.2}$$

The zero coupling limit of $Z(\varepsilon,k)$ can be identified with the symplectic volume of the instanton moduli space $\mathcal{M}_k(g)$ with respect to a Kähler metric g[25]. Of course $\mathcal{M}_k(g)$ is rarely compact. We always assume there are no $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ zero-modes.

We calculate the partition function $Z_{\mathbf{B}}(\varepsilon, k)$ contributed from the branch $\mathbf{B}$ as a simple application of Witten's fixed point theorem [27].[14]

3.1. The fixed points locus

The HYM theory has the same global supersymmetry transformation laws as the $N = 2$ TYM theory. We only deal with the branch $\mathbf{B}$, $\varphi_f \neq 0$. In this branch the BRST fixed point is[15]

$$\varphi_f = \varphi_c T_3 = constant. \tag{3.5}$$

Now we determine the fixed point solution for the gauge connections. They are given by the reducible (holomorphic) connections which, sometimes, will be called the abelian critical points. Consider the space $\mathcal{A}_k$ of all connections of a $SU(2)$-vector-bundle E over X with a given instanton number k. We denote $\mathcal{A}_k^{1,1}$ be the subspace which consists of holomorphic connections of $\mathcal{A}_k$ and $\mathcal{A}_k^*$ be the space of irreducible connections. The space of reducible connections is then $\mathcal{A}_k^{1,1}\backslash\mathcal{A}_k^{*1,1} = \mathcal{A}_k\backslash\mathcal{A}_k^{*1,1}$. A holomorphic connection $A \in \mathcal{A}_k^{1,1}$ endows E with a holomorphic structure $\mathcal{E}_A$. The connection A is reducible if and only if $\mathcal{E}_A$ splits into the sum of holomorphic line bundles

$$\mathcal{E}_A = L_A \oplus L_A^{-1} \tag{3.6}$$

satisfying

$$c_1(L_A)\cdot c_1(L_A) = -k, \tag{3.7}$$

$$H^{1,1}(X;\mathbb{Z}) \times H^{1,1}(X;\mathbb{Z}) \to \mathbb{Z} :$$

---

[14] On the manifold $b_2^+ > 1$ the branch $\mathbf{B}$ is absent generically. This means that the deformation to HYM theory is not particulary useful. We do not how to evaluate the crucial branch $\mathbf{A}$. A careful application of the abelianization techniques [47][53][54] may be used to deal with the non-Abelian branch.

[15] According to our convention of the Tr, the Lie algebra generators are anti-hermitian given by

$$T_a = \frac{\sigma_a}{2i}, \qquad T_\pm = T_1 \pm iT_2, \tag{3.3}$$

with Pauli matrices σ_a and

$$\mathrm{Tr}\, T_a T_b = -\frac{1}{2}\delta_{ab}. \tag{3.4}$$

Since we consider the case $p_g = 0$, the above parings become the intersection parings $q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \to \mathbb{Z}$:

$$\zeta \cdot \zeta = -k, \tag{3.8}$$

where $\zeta \in H^2(X;\mathbb{Z})$. Obviously, the solutions of the above equation are independent of the metrics on X. Furthermore we always obtain in pairs $\pm\zeta$. *For the simply connected case, that we are considering, it is known that the reducible connection corresponding to the pairs $\pm\zeta$ is unique up to gauge equivalence class* [2]. Thus, it is sufficient to solve the above intersection pairings to determine the gauge equivalence class of the abelian critical points of HYM theory. It also follows that the abelian fixed point locus is non-degenerate and isolated set of points. Then, the path integral calculation based on the fixed point theorem gives an exact answer. We will confuse the abelian critical point with its associated line bundle as well as its first Chern class. The value of the curvature two-form $F_f \in \Omega^{1,1}(\mathrm{End}_0(\mathcal{E}_{A_f})$ at the fixed point locus

$$F_f = -2\pi i \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}, \tag{3.9}$$

is determined by the first Chern class of the line bundles ζ satisfying (3.8).

The values of the other fields in the fixed point locus can be read from (2.17) and (2.23)

$$\psi_f = \bar{\psi}_f = H_f^{2,0} = H_f^{0,2} = \chi_f^{2,0} = \bar{\chi}_f^{0,2} = 0. \tag{3.10}$$

Of course, the values of $\chi^{2,0}$ and $\bar{\chi}^{0,2}$ in the fixed point locus need not to be zero. It is sufficient to take values in the Cartan (abelian) sub-algebra to satisfy $\mathbf{s}H^{2,0} = -i[\varphi, \chi^{2,0}] = 0$ or $\bar{\mathbf{s}}H^{2,0} = -i[\varphi, \bar{\chi}^{0,2}] = 0$. Since we are dealing with the manifold satisfying $H^{2,0}(X) = H^{0,2}(X) = 0$, we can set the values zero without loss of generality.

We expand the action functional about the fixed point locus up to the quadratic terms in the action

$$S_H \approx S_f + S^{(2)}, \tag{3.11}$$

where the action S_f in the fixed points is given by the intersection number

$$S_f = \frac{i}{2\pi}\varphi_c(\zeta \cdot H) + \frac{\varepsilon}{32\pi^2}(H \cdot H)\varphi_c^2, \tag{3.12}$$

for every integral cohomology classes $\zeta \in H^2(X;\mathbb{Z})$ satisfying $\zeta \cdot \zeta = -k$.

3.2. **The gauge fixing**

We choose unitary gauge in which

$$\varphi_{\pm} = 0, \tag{3.13}$$

where

$$\varphi = (\varphi_c + \varphi_3)T_3 + \varphi_+ T_+ + \varphi_- T_-. \tag{3.14}$$

In this gauge φ has values on the maximal torus(Cartan sub-algebra).

By following the standard Faddev-Povov gauge fixing, we introduce a new nilpotent BRST operator δ with the algebra

$$\begin{aligned} \delta\varphi_{\pm} = \pm c_{\pm}(\varphi_c + \varphi_3), \qquad \delta c_{\pm} = 0 \\ \delta\varphi_3 = \delta\varphi_c = 0, \qquad \delta\bar{c}_{\pm} = b_{\pm}, \qquad \delta b_{\pm} = 0, \end{aligned} \tag{3.15}$$

where $c_{\pm}$ and $\bar{c}_{\pm}$ are anti-commuting ghosts and anti-ghosts respectively, and $b_{\pm}$ are commuting auxiliary fields. The action for gauge fixing terms reads

$$\begin{aligned} S_{gauge} =& \delta\left[\frac{i}{4\pi^2}\int_X i\big(\bar{c}_-\varphi_+ + \bar{c}_+\varphi_-\big)\right] \\ =& \frac{1}{4\pi^2}\int_X \left[-\big(b_-\varphi_+ + b_+\varphi_-\big) + \bar{c}_-(\varphi_c + \varphi_3)c_+ - \bar{c}_+(\varphi_c+\varphi_3)c_-\right]\frac{\omega^2}{2!}. \end{aligned} \tag{3.16}$$

The resultant action has the δ-BRST symmetry and, hence we can use the fixed point theorem. The fixed point locus for δ-BRST is $c_{\pm} = b_{\pm} = 0$. The Gaussian integrations over $\bar{c}$ and c give

$$\Big[det_{\pm}(\varphi_c)\Big]_{\Omega^0}. \tag{3.17}$$

3.3. **The transverse part (I)**

The transverse part corresponding the $\pm$ part of the Lie algebra can be easily calculated by simple Gaussian integrals.

<u>*Bosonic Sector (doublet i. e. $\pm$ part)*</u>

The quadratic action relevant to this sector is given by

$$
\begin{aligned}
S^{(2)}_{\pm}(bose) =& \frac{i}{h^2}\int_X \Big[H^{2,0}_{+}\wedge *\bar{\partial}_a A''_{-} + H^{2,0}_{-}\wedge *\bar{\partial}_a A''_{+} + H^{0,2}_{+}\wedge *\partial_a A'_{-} + H^{0,2}_{-}\wedge *\partial_a A'_{+} \Big] \\
&+ \frac{i}{4\pi^2}\int_X i\varphi_c \big(A'_{+}\wedge A''_{-} + A''_{+}\wedge A'_{-}\big)\wedge\omega
\end{aligned}
\tag{3.18}
$$

where

$$
\partial_a \alpha_{\pm} = \partial\alpha_{\pm} \pm iA_f \alpha_{\pm} \tag{3.19}
$$

for any doublet $\alpha_{\pm}$.

This is a Gaussian integral which can be evaluated by shifting the variables

$$
\begin{aligned}
A'_{\pm} &\to A'_{\pm} \pm \frac{4\pi^2}{h^2\varphi_c}\partial^*_a H^{2,0}_{\pm} \\
A''_{\pm} &\to A''_{\pm} \mp \frac{4\pi^2}{h^2\varphi_c}\bar{\partial}^*_a H^{0,2}_{\pm}
\end{aligned}
\tag{3.20}
$$

In terms of new variables, we have

$$
\begin{aligned}
S^{(2)}_{\pm}(bose) =& -\frac{4\pi^2 i}{h^4}\int_X \frac{1}{\varphi_c}\Big[\partial^*_a H^{2,0}_{+}\wedge *\bar{\partial}^*_a H^{0,2}_{-} + \bar{\partial}^*_a H^{0,2}_{+}\wedge *\partial^*_a H^{2,0}_{-} \Big] \\
&- \frac{i}{4\pi^2}\int_X \varphi_c \big(A'_{+}\wedge *A''_{-} + A''_{+}\wedge *A'_{-}\big),
\end{aligned}
\tag{3.21}
$$

where we have used the Kähler identities (2.26) as well as the relation [16]

$$
\int_X \alpha^{1,0}\wedge\alpha^{0,1}\wedge\omega = i\int_X \alpha^{1,0}\wedge *\alpha^{0,1}, \tag{3.22}
$$

and self-duality of H. By the Gaussian integrals over A and H's, we have

$$
\Big[det_{\pm}(\varphi_c)\Big]^{-1/2}_{\Omega^{1,0}\oplus\Omega^{0,1}} \cdot \Big[det_{\pm}\big(\frac{\partial_a\partial^*_a}{\varphi_c}\big)\Big]^{-1/2}_{\Omega^{2,0}} \cdot \Big[det_{\pm}\big(\frac{\bar{\partial}_a\bar{\partial}^*_a}{\varphi_c}\big)\Big]^{-1/2}_{\Omega^{0,2}}. \tag{3.23}
$$

The fermionic sector (doublet i. e. $\pm$ part)

[16] This is also known as the Kähler identity. In fact we heavily rely on many special properties of the Kähler manifold in the calculations, which are not valid for non-Kähler case.

We can compute, in a similar fashion, the contribution from the transverse fermion doublets

$$
\begin{aligned}
S^{(2)}_{\pm}(fermi) = &-\frac{i}{h^2}\int_X \Big[\chi^{2,0}_{+}\wedge *\bar\partial_a\bar\psi_{-} + \chi^{2,0}_{-}\wedge *\bar\partial_a\bar\psi_{+} + \bar\chi^{0,2}_{+}\wedge *\partial_a\psi_{-} + \bar\chi^{0,2}_{-}\wedge *\partial_a\psi_{+}\Big] \\
&+\frac{i}{4\pi^2}\int_X \Big[\psi_{+}\wedge *\bar\psi_{-} + \psi_{-}\wedge *\bar\psi_{+}\Big]
\end{aligned}
\tag{3.24}
$$

where we have used (3.22) . After changing variables by

$$
\begin{aligned}
\psi_{\pm} &\to \psi_{\pm} - \frac{4\pi^2 i}{h^2}\partial^*_a\chi^{2,0}_{\pm} \\
\bar\psi_{\pm} &\to \bar\psi_{\pm} - \frac{4\pi^2 i}{h^2}\bar\partial^*_a\bar\chi^{2,0}_{\pm}
\end{aligned}
\tag{3.25}
$$

we have

$$
\begin{aligned}
S^{(2)}_{\pm}(fermi) = &-\frac{4\pi^2 i}{h^4}\int_X \Big[\partial^*_a\chi^{2,0}_{+}\wedge *\bar\partial^*_a\bar\chi^{0,2}_{-} + \partial^*_a\chi^{2,0}_{-}\wedge *\bar\partial^*_a\bar\chi^{0,2}_{+}\Big] \\
&+\frac{i}{4\pi^2}\int_X \Big[\psi_{+}\wedge *\bar\psi_{-} + \psi_{-}\wedge *\bar\psi_{+}\Big]
\end{aligned}
\tag{3.26}
$$

The Gaussian integrals over ψ and χ's give

$$
\Big[det_{\pm}(\partial_a\partial^*_a)\Big]^{1/2}_{\Omega^{2,0}}\cdot\Big[det_{\pm}(\bar\partial_a\bar\partial^*_a)\Big]^{1/2}_{\Omega^{0,2}}.
\tag{3.27}
$$

3.4. The transverse part (II) : $U(1)$ singlets

The remaining quadratic action is given by

$$
\begin{aligned}
S^{(t)}_3 = &\frac{i}{h^2}\int_X \Big[\frac{1}{2}H^{2,0}_3\wedge *\bar\partial A''_3 + \frac{1}{2}H^{0,2}_3\wedge *\partial A'_3 - \frac{1}{2}\chi^{2,0}_3\wedge *\bar\partial\bar\psi_3 - \frac{1}{2}\bar\chi^{0,2}_3\wedge *\partial\psi_3\Big] \\
&+\frac{1}{4\pi^2}\int_X \Big[\frac{i}{2}\varphi_3\big(\partial A''_3 + \bar\partial A'_3\big) + \psi_3\bar\psi_3\Big]\wedge\omega + \frac{\varepsilon}{8\pi^2}\int_X \frac{\omega^2}{2!}\,\frac{1}{2}\varphi^2_3.
\end{aligned}
\tag{3.28}
$$

We will show that the bosonic and fermionic contributions in this part cancel exactly each other and so give trivial result. Note that every field in here doesn't contain any zero mode for ∂ and $\bar\partial$ operator. We explicitly decomposed T_3 component of φ as zero mode φ_c and non-zero mode φ_3 from the beginning and similarly for the $U(1)$ connections as the topologically non-trivial part, which is contained in the action S_f, and trivial one. Furthermore as $h^{1,0} = h^{0,1} = h^{2,0} = h^{0,2} = 0$, where $h^{p,q}$ denotes the Hodge number, in our considerations all other fields do not have any zero-mode[17].

[17] Actually we can remove the condition $h^{1,0} = h^{0,1} = 0$. We leave this as an exercise.

From $H_3^{2,0}$ and $H_3^{0,2}$ integrations, we get the delta function constraints

$$\bar{\partial} A_3'' = 0, \qquad \partial A_3' = 0. \tag{3.29}$$

Similarly the $\chi_3^{2,0}$ and $\bar{\chi}_3^{0,2}$ integrations give the constraints

$$\bar{\partial}\bar{\psi}_3 = 0, \qquad \partial \psi_3 = 0. \tag{3.30}$$

From the Kähler identities (2.26), we have

$$\begin{aligned} \partial A_3' &= \partial^* A_3' = 0, \qquad \partial\psi_3 = \partial^*\psi_3 = 0, \\ \bar{\partial} A_3'' &= \bar{\partial}^* A_3'' = 0, \qquad \bar{\partial}\psi_3 = \bar{\partial}^*\bar{\psi}_3 = 0. \end{aligned} \tag{3.31}$$

This shows that all of them are harmonic one-forms which implies they actually vanishes. Thus the delta function constraints (3.31) and (3.30) are equivalent to the delta function constraint,

$$\prod_{x\in X} \delta(A_3'(x))\delta(A_3''(x))\delta(\psi_3(x))\delta(\bar{\psi}_3(x)).$$

Thus the integrations over $A_3', A_3'', \psi_3, \bar{\psi}_3$ just lead to an unity. Finally, the integration of non-constant mode φ_3 is trivial as only the quadratic term left in the action.

By combining (3.17) , (3.23) and (3.27) , we have the contributions of the transverse degrees of freedom as follows:

$$\begin{aligned} &\Big[det_\pm(\varphi_c)\Big]_{\Omega^0} \cdot \Big[det_\pm(\varphi_c)\Big]^{-1/2}_{\Omega^{1,0}\oplus\Omega^{0,1}} \cdot \Big[det_\pm(\varphi_c)\Big]^{-1/2}_{\Omega^{2,0}\oplus\Omega^{0,2}} \\ =&\Big[det_\pm(\varphi_c)\Big]^{1/2}_{(\Omega^0\ominus\Omega^{1,0}\oplus\Omega^{2,0})\oplus(\Omega^0\ominus\Omega^{0,1}\oplus\Omega^{0,2})} \\ =&\Big[det(\varphi_c)\Big]^{index\bar{\partial}_{(+)}+index\bar{\partial}_{(-)}} = \varphi_c^{2(1-h^{0,1}+h^{0,2})-4k} = \varphi_c^{2-4k}, \end{aligned} \tag{3.32}$$

where we used the index theorem of the twisted Dolbeault operators.

4. The Partition Function

The path integral of the previous section shows that partition function essentially reduces to the following expression

$$Z_{\mathbf{B}}(\varepsilon, k) = \sum_{\zeta_i\cdot\zeta_i=-k} \int_{-\infty}^{\infty} d\varphi \frac{1}{\varphi^{4k-2}} \exp\Big[-\frac{i}{2\pi}\varphi(\zeta_i\cdot H) - \frac{\varepsilon}{32\pi^2}\varphi^2(H\cdot H)\Big], \tag{4.1}$$

where we omitted the superscript c from φ. In this section we examine the small coupling behavior of the above partition function [18] and its variations according to the metric.

4.1. The Small Coupling Behavior

The above equation has pole at $\varphi = 0$, which can be removed by deforming the contour from $\mathbb{R}$ to $P_\pm = \{Im\,\varphi = \pm\kappa\}$ and taking the limit $\kappa \to 0$. Of course the integral should be independent to the contours. For $\varphi \in P_\pm$, we can write

$$\begin{aligned} &Z_{\mathbf{B}}(\varepsilon, k) \\ &\quad = -\sum_{\zeta_i\cdot\zeta_i=-k} \frac{1}{(4k-3)!}\int_0^\infty ds\, s^{4k-3}\int_{P_\pm} d\varphi \exp\left[\pm i\varphi s - \frac{i}{2\pi}\varphi(\zeta_i\cdot H) - \frac{\varepsilon}{32\pi^2}\varphi^2(H\cdot H)\right] \\ &\quad = -\sum_{\zeta_i\cdot\zeta_i=-k} \frac{1}{(4k-3)!}\sqrt{\frac{32\pi^3}{\varepsilon(H\cdot H)}}\int_0^\infty ds\, s^{4k-3}\exp\left[-\frac{8\pi^2}{\varepsilon(H\cdot H)}\left(s\pm\frac{\zeta_i\cdot H}{2\pi}\right)^2\right]. \end{aligned} \tag{4.2}$$

We choose the $+$ sign and decompose the above equation as

$$\begin{aligned} Z_{\mathbf{B}}(\varepsilon, k) = &-\sum_{\zeta_i\cdot H<0}\frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3}e^{-\frac{1}{2\varepsilon'}\left(s+\frac{\zeta_i\cdot H}{2\pi}\right)^2} \\ &-\sum_{\zeta_i\cdot H=0}\frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3}e^{-\frac{1}{2\varepsilon'}s^2} \\ &-\sum_{\zeta_j\cdot H>0}\frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3}e^{-\frac{1}{2\varepsilon'}\left(s+\frac{\zeta_j\cdot H}{2\pi}\right)^2}, \end{aligned} \tag{4.3}$$

where we set $\varepsilon' = \varepsilon(H\cdot H)/16\pi^2$. One can easily see the last term vanishes in the limit

[18] The similar asymtotic estimation of the finite dimensional integral (4.1) was discussed by Wu [46] in his study of the abelian localization as a special case of the non-abelian localization. Note that the branch $\mathbf{B}$ is governed by abelian localization rather than the sophiscated non-abelian localization.

$\varepsilon' \to 0$ up to exponentially small term as follows;

$$\begin{aligned}
&\sum_{\zeta_i\cdot H>0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'}\left(s+\frac{\zeta_i\cdot H}{2\pi}\right)^2} \\
&\quad\le \sum_{\zeta_j\cdot H>0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'}\left(s^2+\left(\frac{\zeta_j\cdot H}{2\pi}\right)^2\right)} \\
&\quad= \sum_{\zeta_j\cdot H>0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'}s^2}\cdot e^{-\frac{1}{\varepsilon}\cdot\frac{2(\zeta_j\cdot H)^2}{H\cdot H}} \\
&\quad= O\left(e^{-\frac{\delta^2}{\varepsilon}}\right),
\end{aligned} \tag{4.4}$$

where $\delta^2 = Min_j(\frac{2(\zeta_j\cdot H)^2}{H\cdot H})$. Similarly we can extract the polynomial dependency on ε, apart from the exponentially small terms, from the first term;

$$\begin{aligned}
&-\sum_{\zeta_i\cdot H<0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'}\left(s+\frac{\zeta_i\cdot H}{2\pi}\right)^2} \\
&\quad= -\sum_{\zeta_j\cdot H<0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_{-\infty}^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'}\left(s+\frac{\zeta_j\cdot H}{2\pi}\right)^2} + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right) \\
&\quad= -\sum_{\zeta_j\cdot H<0} \frac{1}{(4k-3)!}\sqrt{\frac{2\pi}{\varepsilon'}}\int_{-\infty}^\infty ds' \left(s'-\frac{\zeta_j\cdot H}{2\pi}\right)^{4k-3} e^{-\frac{1}{2\varepsilon'}s'^2} + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right) \\
&\quad= \sum_{\zeta_i\cdot H<0}\sum_{r=0}^{[d/2]} \frac{2\pi}{(d-2r)!\,r!\,2^r}\left(\frac{\zeta_i\cdot H}{2\pi}\right)^{d-2r}(\varepsilon')^r + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right) \\
&\quad= \frac{1}{(2\pi)^{d-1}}\sum_{\zeta_i\cdot H<0}\sum_{r=0}^{[d/2]} \frac{\varepsilon^r}{(d-2r)!\,r!\,2^{3r}}(\zeta_i\cdot H)^{d-2r}(H\cdot H)^r + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right),
\end{aligned} \tag{4.5}$$

where $d = 4k-3$. Assume that there are no reducible ASD connections, i.e., no line bundle ζ_i with $\zeta_i\cdot H = 0$, such that the second term absent. Then, our result for $Z'(\varepsilon,k)$ is

$$Z_{\mathbf{B}}(\varepsilon,k) = \frac{1}{(2\pi)^{d-1}}\sum_{\zeta_i\cdot H<0}\sum_{r=0}^{[d/2]} \frac{\varepsilon^r}{(d-2r)!\,r!\,2^{3r}}(\zeta_i\cdot H)^{4k-3-2r}(H\cdot H)^r + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right), \tag{4.6}$$

where the summation runs over every divisor satisfying $\zeta_i\cdot\zeta_i = -k$ and $\zeta_i\cdot H<0$. If we choose the $-$ sign in the beginning we have the same pattern of the asymptotic behaviors and the divisors satisfying $\zeta_i\cdot H>0$ only contributes. Since the solutions of $\zeta_i\cdot\zeta_i = -k$ always arise in pairs $\pm\zeta_i$, we have the same result.

4.2. The Variation of the Partition Function

We have seen that the Abelian critical points of HYM theory are the reducible connections which can be uniquely determined by the 2-dimensional integral classes (or line bundles) ζ_i satisfying $\zeta_i \cdot \zeta_i = -k$. Clearly, the notion of the reducible connections is metric independent. On the other hand, the notion of the reducible ASD connections is evidently metric dependent. An Abelian critical point ζ is ASD if and only if $\zeta \cdot H = 0$ where H is Poincaré dual to the Kähler form associated with the given metric. Equivalently, a reducible connection A is ASD if and only if the degree of the associated holomorphic line bundle ζ is zero,

$$\deg(\zeta) = \int_X c_1(\zeta) \wedge \omega = \zeta \cdot H = \frac{i}{2\pi} \int_X F_c \wedge \omega = \frac{i f_c(A)}{2\pi} \int_X \omega \wedge \omega. \tag{4.7}$$

The degree depends only on the cohomology classes of the first Chern class and of the Kähler form. The value $f_c(A_f)$ of the critical points $d_A f_c(A) = 0$ is also determined by the degree and the self-intersection number of the Kähler form. Note that the critical points of the moment map $\mathfrak{m}(A)$ are also determined by the same data. We call a reducible connection as a higher critical point if its degree is non-zero.

Before going on, we should remark that the structure of the abelian critical points is isomorphic to the chamber structure determined by the system $\mathcal{W}_k \subset \overline{\mathcal{W}_k}$ of walls in the positive cone. In particular, for each line bundle associated with an abelian critical point ζ, there is an associated wall W_ζ in the positive cone. Let H_+ and H_- be ample divisor lying in the two chambers C_+ and C_- separated by the single wall W_ζ. We assume that ζ has negative degree with respect to ample divisor H_+ lying in the chamber C_+, i.e., $\zeta \cdot H_+ < 0$. Let H_0 denote ample divisor lying on the wall. If we change the metric, such that the ample divisor H starts from C_+, crosses the wall and goes to the chamber C_-, we have[19]

$$\zeta \cdot H_+ < 0, \qquad \zeta \cdot H_0 = 0, \qquad \zeta \cdot H_- > 0. \tag{4.8}$$

Now it is clear what we have actually done by deforming TYM theory to HYM theory. We mapped all the possible reducible instantons, which appear as we vary the Kähler metric in every possible way, to the abelian critical points of the HYM theory.

[19] The walls can intersect with each other, we assume that our path of metric does not cross the intersection region.

In the limit $\varepsilon \to 0$, the partition function of HYM theory is such that we sum up only critical points with negative degree. As one can easily see, we lost a critical point ζ with negative degree by passing through the wall W_ζ. On the other hand, there is also another abelian critical point $-\zeta$ which defines the same wall W_ζ with ζ. Thus, we get a new critical point $-\zeta$ of negative degree instead. We will collectively denote the ample classes lying in a $C_\pm$ by $H_\pm$. We change the metric such that our ample class H crosses just *one wall* W_ζ, defined by a certain divisor ζ satisfying $\zeta \cdot H_+ < 0$ and $\zeta \cdot \zeta = -k$, and moves to another chamber C_-. Then we have $\zeta \cdot H_- > 0$ by definition. This also amounts that no other line bundles change the sign of their degrees. The contribution of ζ to the partition function $Z_{\mathbf{B}}(\varepsilon, k)(C_+)$ is given by

$$\frac{1}{(2\pi)^{4k-4}} \sum_{r=0}^{[d/2]} \frac{\varepsilon^r}{(d-2r)!\, r!\, 2^{3r}} (\zeta \cdot H_+)^{4k-3-2r} (H_+ \cdot H_+)^r + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right), \tag{4.9}$$

while $Z_{\mathbf{B}}(\varepsilon, k)(C^-)$ no longer receives contributions from ζ. On the other hand, $Z_{\mathbf{B}}(\varepsilon, k)(C_-)$ receives contributions from $-\zeta$, since $-\zeta \cdot H_- < 0$, given by

$$\frac{1}{(2\pi)^{4k-4}} \sum_{r=0}^{[d/2]} \frac{\varepsilon^r}{(d-2r)!\, r!\, 2^{3r}} (-\zeta \cdot H_-)^{4k-3-2r} (H_- \cdot H_-)^r + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right), \tag{4.10}$$

Without loss of generality, we will use a normalization $H \cdot H = 1$, i.e., $H_+ \cdot H_+ = H_- \cdot H_- = 1$.

4.3. The non-analytic part

Up to now, we have considered the case that the metric does not admit reducible ASD connections. Now we allow the Kähler metric to lie on one of the walls $\mathcal{W}_k$ such that there is a reducible ASD connection. We define the multiplicity m of the reducible instanton by the number of the walls intersect at the point where the Kähler metric lies on. Then the partition function (3.2) contains a non-analytic term which is not a polynomial of ε. Now the partition function (4.6) has additional contribution given by

$$\begin{aligned} &- \sum_{\zeta_i \cdot H = 0} \frac{1}{(4k-3)!} \sqrt{\frac{2\pi}{\varepsilon'}} \int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'} s^2} \\ &= - \frac{m}{(4k-3)!} \sqrt{\frac{2\pi}{\varepsilon'}} \int_0^\infty ds\, s^{4k-3} e^{-\frac{1}{2\varepsilon'} s^2} \\ &= - \frac{(2\pi)^{1/2} m}{(4k-3)!!} (\varepsilon')^{2k-3/2}, \end{aligned} \tag{4.11}$$

leading to the modified partition function

$$
\begin{aligned}
Z'(\varepsilon,k) =& \frac{1}{(2\pi)^{4k-4}} \sum_{\zeta_i\cdot H<0} \sum_{r=0}^{[d/2]} \frac{\varepsilon^r}{(d-2r)!\, r!\, 2^{3r}} \left(\zeta_i\cdot H\right)^{4k-3-2r} \\
& - \frac{1}{(2\pi)^{4k-4}} \frac{m}{(2\pi)^{1/2}(4k-3)!!} (\varepsilon/2)^{2k-3/2} \\
& + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right).
\end{aligned}
\tag{4.12}
$$

Our calculation precisely shows that the non-analytic term is the contribution of the reducible instanton. The similar phenomenon has been observed in the two-dimensional version of the TYM theory by Witten.[20]

5. The Relations with the TYM Theory

Now we are going to extract the expectation values of topological observables in the TYM theory. To obtain the precise relations with the TYM theory when we vary the metric, it is conceptually more clear and convenient to use the expectation values of the topological observables rather than the partition function itself. The HYM theory has the same set of topological observables (2.32)(2.33) with the TYM theory. The relation (2.44) between certain correlation function of TYM and the partition of HYM theory can be generalized [26][25]. Picking two-dimensional classes $\Sigma_i \in H_2(X;\mathbb{Z})$, the expectation value $\left\langle \prod_{i=1}^{d} \mu(\Sigma_i) \right\rangle'$ evaluated in the HYM in the limit $\varepsilon \to 0$ is related to that of TYM theory by the formula

$$
\left\langle \prod_{i=1}^{d} \mu(\Sigma_i) \right\rangle' = \left\langle \prod_{i=1}^{d} \mu(\Sigma_i) \right\rangle + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right). \tag{5.1}
$$

This can be further generalized to include the four-dimensional class Θ

$$
\left\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i) \right\rangle' = \sum_{2m+n=2r} \frac{\varepsilon^m}{m!n!} \left\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \tilde{\omega}^n \Theta^m \right\rangle + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right). \tag{5.2}
$$

Thus it is sufficient to calculate (5.2) to determine the general expectation values of TYM theory for a simply connected manifold.[21]

---

[20] We refer the readers to the remarks in Sect. 4.4 of ref. [26].

[21] Actually, it is sufficient to determine the term of power ε^r in the RHS of (5.2).

The expectation value of the observables in HYM theory can be also written as the sum of the contributions of the two branches **A** and **B**;

$$\left\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i) \right\rangle' = \left\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i) \right\rangle'_{\mathbf{A}} + \left\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i) \right\rangle'_{\mathbf{B}} . \tag{5.3}$$

We should emphasize here that, in the HYM theory, all the terms that do not vanish exponentially must be interpreted as the contributions of the ASD connections.[22] That is, $< \cdots >'_{\mathbf{B}}$ for the branch **B** is also the contributions of ASD connections up to the exponentially small terms for $\varepsilon \to 0$. Clearly, $< \cdots >'_{\mathbf{A}}$ for the branch **A** does not contain any exponentially small term. Consequently, we can divided any expectation value of TYM theory as

$$\langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle = \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle_{\mathbf{A}} + \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle_{\mathbf{B}} \tag{5.4}$$

where

$$\begin{aligned} \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle'_{\mathbf{A}} &= \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle_{\mathbf{A}}, \\ \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle'_{\mathbf{B}} &= \langle \prod_{i=1}^{d-2r} \mu(\Sigma_i)\, \Theta^r \rangle_{\mathbf{B}} + O\left(e^{-\frac{\delta^2}{\varepsilon}}\right). \end{aligned} \tag{5.5}$$

If we consider manifold with $b_2^+ \geq 3$, the contribution of the branch **B** is generically absent for both HYM and TYM theories.

In Sect. 5.1, the contribution of the branch **B** to the expectation values is evaluated using the similar method as for the partition function $Z_{\mathbf{B}}(\varepsilon, k)$. However, we do not know how to calculate the contribution of the branch **A**. It is quite natural to believe that the Seiberg-Witten monopole invariants correspond to the branch **A**. In Sect. 5.2, we briefly review some known properties of the Seiberg-Witten invariants for manifolds with $b_2^+ = 1$. Then, we study the variation of the expectation values of TYM theory in Sect. 5.3.

5.1. The branch B

As usual, we choose a generic metric g which does not admit the zero-modes of $\bar{\chi}^{0,2}$. That is, the second instanton cohomology group is trivial. Then the only source of the singularities in the moduli space of ASD connection $\mathcal{M}_k(g)$ is the reducible instantons

22 This statement is originated from [26].

The value $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l) \right\rangle'_{\mathbf{B}}$ for $r = 0, 1, \ldots, [d/2]$ can be evaluated using the fixed point theorem as the partition function $Z_{\mathbf{B}}(\varepsilon, k)$ is. The result can be readily obtained by the effective partition function (4.1) as

$$\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l) \right\rangle'_{\mathbf{B}} = \sum_{\zeta_i \cdot \zeta_i = -k} \prod_{l=1}^{d-2r} \left(-\frac{i(\zeta_i \cdot \Sigma_l)}{2\pi} \right) \int_{-\infty}^{\infty} d\varphi \frac{1}{\varphi^{2r+1}} \exp\left[-\frac{i}{2\pi}\varphi(\zeta_i \cdot H) - \frac{\varepsilon}{32\pi^2}\varphi^2 \right] \tag{5.6}$$

We can study the $\varepsilon \to 0$ limit using the methods in Sect. 4.1.,

$$\begin{aligned} \left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l) \right\rangle'_{\mathbf{B}} &= \sum_{\zeta_i \cdot H < 0} \prod_{l=1}^{d-2r} \left(\frac{\zeta_i \cdot \Sigma_l}{2\pi} \right) \frac{1}{(2r)!} \sqrt{\frac{2\pi}{\varepsilon'}} \int_0^\infty ds\, s^{2r} e^{-\frac{1}{2\varepsilon'}\left(s + \frac{\zeta_i \cdot H}{2\pi}\right)^2} \\ &+ \sum_{\zeta_i \cdot H = 0} \prod_{l=1}^{d-2r} \left(\frac{\zeta_i \cdot \Sigma_l}{2\pi} \right) \frac{1}{(2r)!} \sqrt{\frac{2\pi}{\varepsilon'}} \int_0^\infty ds\, s^{2r} e^{-\frac{1}{2\varepsilon'} s^2} \\ &+ \sum_{\zeta_j \cdot H > 0} \prod_{l=1}^{d-2r} \left(\frac{\zeta_i \cdot \Sigma_l}{2\pi} \right) \frac{1}{(2r)!} \sqrt{\frac{2\pi}{\varepsilon'}} \int_0^\infty ds\, s^{2r} e^{-\frac{1}{2\varepsilon'}\left(s + \frac{\zeta_j \cdot H}{2\pi}\right)^2}, \end{aligned} \tag{5.7}$$

The third term above vanishes exponentially fast for $\varepsilon \to 0$.

It is amusing to note that the second term in (5.7), corresponding to the contributions of the reducible ASD connections, always vanish. If we have a solution ζ for $\zeta \cdot \zeta = -k$ and $\zeta \cdot H = 0$, we always have another solution $-\zeta$. Since $d - 2r = 4k - 3 - 2r$ is always an odd integer, their contributions cancel with each other. Thus, $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l) \right\rangle'_{\mathbf{B}}$ is well-defined even in the presence of the reducible ASD connections.

The first term can be calculated in the same way as (4.5), which gives

$$\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l) \right\rangle'_{\mathbf{B}} = \frac{1}{(2\pi)^{d-1}} \sum_{\zeta_i \cdot H < 0} \frac{\varepsilon^r}{2^{3r}\, r!} \prod_{l=1}^{d-2r} (\zeta_i \cdot \Sigma_l) + \text{lower orders in } \varepsilon + O\left(e^{-\frac{\delta^2}{\varepsilon}} \right). \tag{5.8}$$

From the relation (5.2), (5.4) and (5.5), we have

$$\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{\mathbf{B}} = \frac{1}{(2\pi)^{d-1}} \sum_{\substack{\zeta_i \cdot H < 0, \\ \zeta_i \cdot \zeta_i = -k}} \frac{1}{2^{3r}} \prod_{l=1}^{d-2r} (\zeta_i \cdot \Sigma_l). \tag{5.9}$$

Then, the general form of the expectation values of the topological observables of TYM theory is given by

$$\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle = \left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{\mathbf{A}} + \frac{1}{(2\pi)^{d-1}} \sum_{\substack{\zeta_i \cdot H < 0, \\ \zeta_i \cdot \zeta_i = -k}} \frac{1}{2^{3r}} \prod_{l=1}^{d-2r} (\zeta_i \cdot \Sigma_l). \tag{5.10}$$

5.2. **The Seiberg-Witten invariants**

One of the open problem in our approach is to uncover the genuine diffeomorphism invariants. This is because we computed the path integral only for one of the two branches of the fixed points. Clearly, the branch **B**, which we have calculated, is generically absent for manifold with $p_g > 1$. For the fixed point $\varphi = 0$ (the branch **A**), we have the full $SU(2)$ symmetry and the path integral has contribution from the irreducible instantons. There is no reason that the contribution would vanish. Unfortunately, the path integral for this branch reduces to a formal expression such as one integral over the moduli space of irreducible instantons. Thus, we need an alternative approach to deal with this branch.

Note that the limit $t \to 0$ in eq.(2.29) can also be viewed as the limit $h^2 \to \infty$ with t fixed. In this limit, the semi-classical analysis is invalidated. At this point we can utilize the fundamental results of Seiberg-Witten on the strong coupling behaviour of the untwisted $N = 2$ super-Yang-Mills theory [28][29][30]. Their result can be essentially summarized by the quantum moduli space parametrized by a complex variable u, which corresponds to the observable Θ in the twisted theory. Classically, there is a singularity at the origin $u = 0$ where the full $SU(2)$ symmetry is restored. Quantum mechanically, the complex u-plane has two singularities at $u = \pm 1$. The singularities in that plane represent the appearance of new massless particles. For manifolds with $p_g > 1$ one can introduce a perturbation utilizing holomorphic two-forms such that the only contribution comes from the two singular points. Such an effective low energy theory turns out to be an $N = 2$ super-Maxwell theory coupled with hyper-multiplet. One can twist this theory and the resulting theory gives the dual description of the Donaldson invariants[30].[23]

For manifold with $p_g = 0$, no such perturbation is possible and one should integrate over the whole complex plane [30].[24] The extra contributions from the generic points

[23] The Seiberg-Witten monopole equation is the close cousin of the equation appeared in the Vafa and Witten's paper on a twisted $N = 4$ super-Yang-Mills theory [23]. The simlarity has an obvious origin since the $N = 4$ theory can be viewed as an $N = 2$ theory coupled with a $N = 2$ matter multiplet in the adjoint represenation. The similar equation also appears in [55], although we do not know the origin of the similarity. We would like to mention that the general $N = 2$ super-Yang-Mills theory with hypermultiplets can be twisted to define a set of topological field theories which lead to certain non-abelian version of the Seiberg-Witten monopole equation [56].

[24] It was noticed that the Donaldson invariants and Seiberg-Witten invariants are very different for manifold with $b_2^+ = 1$. There is no mystery in it since Seiberg-Witten monopole invariant is one of two parts of the Donaldson invariant.

of the quantum moduli space are the contributions of the abelian instantons. What we have calculated in this paper is essentially such contributions. We can view the original Seiberg-Witten monopole invariants as the contribution of the branch **A**.

The Seiberg-Witten invariants can be viewed as the pairs (x, n_x) where $x \in H^2(X;\mathbb{Z})$ with $x \equiv w_2(X)$ mod 2 and n_x is an integer associated with x. For each x we have an associated holomorphic line bundle L such that $x = -c_1(L^2) = -2c_1(L)$. For each L we have the Seiberg-Witten monopole equation. The dimension of the moduli space $\mathcal{M}_{SW}$ of that monopoles is given by

$$dim_{\mathbb{R}}\mathcal{M}^x_{SW} = W_x = \frac{x\cdot x - (2\chi + 3\sigma)}{4}. \tag{5.11}$$

The amazing fact is that the moduli space is compact. If $W_x = 0$ such that $x^2 = 2\chi + 3\sigma$, one simply counts the number of the points with sign according to a suitable orientation. The integer n_x is that algebraic sum of the points. If $W_x \neq 0$ and $W_x = 2a$, one defines

$$n_x = \int_{\mathcal{M}^x_{SW}} \nu(\Sigma)^a, \tag{5.12}$$

where $\nu : H_2(X;\mathbb{Z}) \to H^2(\mathcal{M}_{SW};\mathbb{Z})$ analogous to the Donaldson μ-map. We would like to remark the followings.

i) The Seiberg-Witten invariants are metric independent for manifold with $b_2^+ > 1$. Witten completely determined the invariants for Kähler surface with $b_2^+ > 1$. For Kähler surface with $p_g > 1$, all the higher dimensional Seiberg-Witten invariants vanish (the simple condition). On the other hand, it is not known if such a property still hold for manifold with $b_2^+ = 1$.

ii) The Seiberg-Witten invariants vanish if a metric admits positive scalar curvature. It is easy to see that the Kähler surface with $p_g > 1$ does not admit a metric of positive curvature except for the hyperKähler case. So there are at least two Seiberg-Witten invariants including $(K_X, 1)$ and $(-K_X, -1)$.

iii) The precise relation between the Seiberg-Witten invariants and the Donaldson invariants for manifold of simple type with $b_2^+ \geq 3$ has been established by Witten [30]. The Seiberg-Witten class x corresponds to the basic class of Kronheimer-Mrowka and n_x to the coefficient associated to the basic class [57]. The only other informations in the Donaldson invariant are all homotopy invariants, such as the intersection form.

Now we come back to our original problem for a simply connected Kähler surface with $p_g = 0$. Similarly to the Donaldson invariants, the Seiberg-Witten invariants (x, n_x) is a topological invariants if n_x does not change for a smooth path of metric joining two generic metrics. It turns out n_x jumps by ± 1 if there is a metric such that the monopole equation reduces to the equation for the abelian instanton [30][58][59]. This amounts to

$$\int_X x \wedge \omega_g = 0. \tag{5.13}$$

Since ω_g belongs to the positive cone, the above can happen if and only if $x \cdot x < 0$, as we discussed before. Conversely, n_x is metric independent if $x \cdot x \geq 0$.

A necessary condition that n_x can be non-vanishing is the non-negativity of the dimension W_x,

$$x \cdot x \geq 2\chi + 3\sigma. \tag{5.14}$$

Using $2\chi + 3\sigma = 9 - n$ for manifold of type $(1, n)$ with $H^1(X; \mathbb{R}) = 0$, one can conclude that all the Seiberg-Witten invariants are metric independent for manifold with $n \leq 9$ (or equivalently $b_2^- \leq 9$).

Now consider the manifold of type $(1, 9+N)$ with $N > 0$. Any possibly non-vanishing, at least for certain metric, Seiberg-Witten class x with $x \equiv w_2(X)$ mod 2 should satisfy

$$x \cdot x \geq -N. \tag{5.15}$$

One can divide the set of all such x into $\{x_{inv}\} \cup \{x'\}$ where $x_{inv} \cdot x_{inv} \geq 0$. Clearly $n_{x_{inv}}$ is metric independent. One can easily show that all the $n_{x_{inv}}$ are identically vanish. Recall that all n_x vanish for any x with a metric which admits positive scalar curvature. From $c_1(X) \wedge \omega_g = -K_X \wedge \omega_g = \frac{1}{2} R_g \omega_g \wedge \omega_g$ where $\omega_g \wedge \omega_g$ is positive definite, we have

$$R_g < 0, \quad \text{for} \quad K_X \cdot [\omega_g] > 0. \tag{5.16}$$

Assume that X has a metric g with negative scalar curvature, i.e, $K_X \cdot [\omega_g] > 0$. Since $K_X \cdot K_X = -N < 0$, there is a period point $[\omega_{g'}]$ in the positive cone for some metric g' such that $[\omega_{g'}] = K_X^{\perp}$, i.e., $K_X \cdot [\omega_{g'}] = 0$. Furthermore there exists a metric g'' such that $K_X \cdot [\omega_{g''}] < 0$. Thus the positive cone always has a chamber whose associated metrics admit positive scalar curvature. It follows that all n_x vanish at that chamber. Since

$n_{x_{inv}}$ is metric independent it should vanish identically. Consequently, every possibly non-vanishing x at least for certain metric satisfies

$$-N \leq x \cdot x \leq -1 \qquad \text{and} \qquad x \equiv w_2(X) \bmod 2. \tag{5.17}$$

Now we can conclude that for manifold of type $(1, 9+N)$ there are no Seiberg-Witten class which are independent of the metric.[25] Similarly to the Donaldson invariants, the Seiberg-Witten invariant depends only on the chamber structure defined by x satisfying (5.17) and $x \equiv w_2(X) \bmod 2$.

Consider one parameter family of g_t joining two generic metrics g_{-1} and g_1 with g_o such that $[\omega_{g_0}] \in x^{\perp}$ and g_t cross the wall $x^{\perp}$ transversely. Witten showed that

$$n_x(g_{-1}) = n_x(g_1) \pm 1. \tag{5.18}$$

5.3. The expectation values in TYM theory

At present we do not know the precise relation between the Seiberg-Witten invariants and the expectation values $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{\mathbf{A}}$. In any case, $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{\mathbf{A}}$ should be, in general, polynomials of degree d in $H_2(X;\mathbb{Z})$ depending on (x, n_x) and q_X. Since the values at the both two branches are well-defined, we can confirm Witten's original claim that the path integral approach is well-defined whatever properties the moduli space of ASD connections has [8].

If we consider manifold of type $(1, n)$ with $n \leq 9$, the variation of the expectation value (5.10) of TYM theory comes only from the branch **B**. The relevant chamber structure in this case is the set $\mathcal{C}_X^k$ of the connected components in the positive cone after removing the system $\mathcal{W}_k$ of walls defined by all the two-dimensional integral classes ζ_i satisfying $\zeta_i \cdot \zeta_i = -k$. The expectation values $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle$ depend on metric only by the chamber structure $\mathcal{C}_X^k$. Let C_+ and C_- be the two chambers separated by the single wall W_ζ with the same condition as (4.8). From (5.10), we have

$$\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{C_+} - \left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\, \Theta^r \right\rangle_{C_-} = \frac{2}{(2\pi)^{d-1}} 2^{3r} \prod_{l=1}^{d-2r} (\zeta \cdot \Sigma_l). \tag{5.19}$$

For a manifold of type $(1, 9+N)$ with $N > 0$, the general transition formula can be more complicated. We should consider the both chamber structures for the branches **A** and **B**. Assuming that the metric crosses a single wall W_ζ which does not intersect with the walls $x^{\perp}$ defined by x satisfying (5.17), the transition formula is given by (5.19).

[25] This does not necessarily mean that there are not genuine diffeomorphism invariants.

6. The Relations with the Donaldson Invariants

In this section, we discuss some conjectural relations between the expectation values of TYM theory and the corresponding Donaldson invariants. Clearly, the expectation value (5.10) does not coincide to the corresponding Donaldson invariants $\overline{q}_{k,X}(\Sigma_1,\ldots,\Sigma_{d-2r},(pt)^r)$. Throughout this paper, we have tried to convince the (especially the mathematician) readers that the path integral approach to the gauge theoretic invariants are well-defined whatever properties the moduli space of ASD connections has. It is clear, due to Witten, that the TYM theory correctly determines the Donaldson invariants at least for manifold with $b_2^+ \geq 3$. For a manifold with $b_2^+ = 1$, the expectation values of TYM theory in general have additional contributions essentially due to the reducible connections. We have shown that the expectation values are well-defined and those additional contributions can be determined exactly. In any case, the path integral approaches do not refer to the compactification procedure to get well-defined results.

Now the underlying reason for the discrepancy between the expectation values of TYM theory and the Donaldson invariants is clear. If we incorporated compactification of the moduli space, the path integral will receive additional contributions from the reducible connections with lower instanton numbers. Similarly to the expectation value of the TYM theory, one can divide the Donaldson invariant into the sum of two branches

$$\overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r) = \overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r)_{\mathbf{A}} + \overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r)_{\mathbf{B}}. \quad (6.1)$$

The equivalence between the TYM theory and Donaldson theory for the manifolds with $b_2^+ \geq 3$ implies that

$$\overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r)_{\mathbf{A}} = \left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\,\Theta^r \right\rangle_{\mathbf{A}}. \quad (6.2)$$

Then the remaining part $\overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r)_{\mathbf{B}}$ should be expressed as the sum of contributions of reducible holomorphic connections with the instanton numbers $1,2,....,k$. In particular, the expression $\left\langle \prod_{l=1}^{d-2r} \mu(\Sigma_l)\,\Theta^r \right\rangle_{\mathbf{B}}$ can be regarded as the contribution of the reducible holomorphic connections with the instanton number k;

$$\overline{q}_{k,X}(\Sigma_1,..,\Sigma_{d-2r},(pt)^r)_{\mathbf{B}} = \frac{1}{2}\sum_{\substack{\zeta_i\cdot H<0,\\ \zeta_i\cdot\zeta_i=-k}} \frac{1}{2^{3r}}\prod_{l=1}^{d-2r}(\zeta_i\cdot\Sigma_l) + \ldots, \quad (6.3)$$

where we normalized the expectation value by multiplying an universal factor

$$(-1)^k \frac{(2\pi)^{4k-4}}{2}.$$

Throughout this section, we always assume that the path of metric does not cross the walls responsible for the changes of the Seiberg-Witten invariants. We change the metric such that our ample class H cross just one wall W_ζ, defined by a certain divisor ζ satisfying $\zeta \cdot H_+ < 0$ and $\zeta \cdot \zeta = -k$, and move to another chamber C_- with $\zeta \cdot H_- > 0$. Provided that the path of metric does not cross another walls defined by divisors ζ' satisfying $\zeta' \cdot \zeta' = -1, ..., -k+1$, we immediately have the transition formula

$$\overline{\Gamma}_X^{k,r}(C_+) - \overline{\Gamma}_X^{k,r}(C_-) = (-1)^k \frac{1}{2^{3r}} \zeta^{d-2r}. \tag{6.4}$$

or

$$\overline{q}_{X,k,C_+}(\Sigma^{d-2r}, (pt)^r) - \overline{q}_{X,k,C_-}(\Sigma^{d-2r}, (pt)^r) = (-1)^k \frac{1}{2^{3r}} (\zeta \cdot \Sigma)^{d-2r}. \tag{6.5}$$

The transition formula (6.4) is a generalization of the formulas obtained by Mong [6] and Kotschick [5] for $r = 0$ by Friedman and Qin [33] for the case $r = 1$. It also agrees with the results of Ellingsrud and Göttsche [34]for general values of r up to certain normalization difference.

6.1. **The problem associated with the compactification**

Now the question is to determine the contribution of the reducible connections from the lower stratas in the compactified moduli space. If we compactify the moduli space, the path integral will get additional contributions from the reducible critical points whose associated divisors satisfying $-k+1 \leq \zeta \cdot \zeta < 0$. Thus, we will simply view our result as the contribution from the top strata. To determine the extra contributions, at least partially, we will use the recent result of Hu and Li [36]which shows that

$$\overline{\mathcal{M}}_k = \bigcup_{\ell=0}^{k-1} \mathcal{M}_{k-\ell} \times \mathrm{Sym}^\ell(X), \tag{6.6}$$

if k is sufficiently large (the condition that the Kähler metrics on X behave as generic metrics).

We consider a d-dimensional subspace $\mathcal{N}_{k-\ell} = N_{k-\ell} \times \mathrm{Sym}^\ell(X) \subset \overline{\mathcal{M}}_k$ such that $\mathcal{M}_{k-\ell} \subset N_{k-\ell}$, $\dim_{\mathbb{C}}(N_\ell) = d - 2\ell$. We also assume that $N_{k-\ell} \backslash \mathcal{M}_{k-\ell}$ does not contains any reducible connections, for arbitrary metric, other than $\mathcal{M}_{k-\ell} \backslash \mathcal{M}^*_{k-\ell}$. We consider $\bigcup_{\ell=0}^k \mathcal{N}_\ell \subset \overline{\mathcal{M}}_k$. The restriction of $\overline{\mu}(\Sigma)$ to $N_{k-\ell} \times \mathrm{Sym}^\ell(X)$ will be in the form

$$\overline{\mu}(\Sigma)_\ell = \mu(\Sigma)_\ell + 2H', \tag{6.7}$$

where $H' \in H^2(\mathrm{Sym}^\ell(X))$. We can consider the part of the Donaldson invariants contributed from $\mathcal{N}_{k-\ell}$,

$$\begin{aligned} \left\langle \overline{\mu}(\Sigma)^d, \mathcal{N}_{k-\ell} \right\rangle &= 2^{2\ell} \frac{d!}{(d-2\ell)!(2\ell)!} \left\langle \mu(\Sigma)_\ell^{d-2\ell}, N_{k-\ell} \right\rangle \left\langle (H')^{2\ell}, \left[\mathrm{Sym}^\ell(X)\right] \right\rangle \\ &= 2^\ell \frac{d!}{(d-2\ell)!\ell!} \left\langle \mu(\Sigma)_\ell^{d-2\ell}, N_{k-\ell} \right\rangle q^\ell, \end{aligned} \tag{6.8}$$

where we have used

$$\left\langle (H')^{2\ell}, \left[\mathrm{Sym}^\ell(X)\right] \right\rangle = \frac{(2\ell)!}{2^\ell \ell!} q^\ell, \text{ where } q \in \mathrm{Sym}^2(H^2(X;\mathbb{Z})). \tag{6.9}$$

Now we can use our previous result to determine $\left\langle \mu(H)_\ell^{d-2\ell}, N_{k-\ell} \right\rangle_{\mathbf{B}}$, which gives

$$\frac{1}{2} \sum_{\zeta_i \cdot H < 0} (-1)^{k-\ell} \left(\zeta_i \cdot \Sigma\right)^{d-2\ell}. \tag{6.10}$$

where the summation runs over every divisor satisfying $\zeta_i \cdot \zeta_i = -k + \ell$. By summing up, we can write

$$\overline{q}_{X,k,C(X)}(\Sigma^d)_{\mathbf{B}} = \frac{1}{2} \sum_{\ell=0}^k (-1)^{k-\ell} 2^\ell \sum_{\substack{\zeta_i \cdot H < 0, \\ \zeta_i \cdot \zeta_i = -k+\ell}} \frac{d!}{(d-2\ell)!\,\ell!} \left(\zeta_i \cdot \Sigma\right)^{d-2\ell} q^\ell + \dots \tag{6.11}$$

In other words

$$\overline{\Gamma}^k_X(C)_{\mathbf{B}} = \frac{1}{2} \sum_{\ell=0}^k (-1)^{k-\ell} 2^\ell \sum_{\substack{\zeta_i \cdot H < 0, \\ \zeta_i \cdot \zeta_i = -k+\ell}} \frac{d!}{(d-2\ell)!\,\ell!} (\zeta_i)^{d-2\ell} q^\ell + \dots \tag{6.12}$$

Let a smooth path of Kähler metric meet only one wall W_ζ defined by ζ such that

$$\zeta \cdot \zeta = -k + \ell_\zeta, \qquad \zeta \cdot H_+ < 0 < \zeta \cdot H_-. \tag{6.13}$$

We have

$$\overline{\Gamma}^k_X(C_+) - \overline{\Gamma}^k_X(C_-) = (-1)^{k-\ell_\zeta} 2^{\ell_\zeta} \frac{d!}{(d-2\ell_\zeta)!\,\ell_\zeta!} (\zeta)^{d-2\ell_\zeta} q^{\ell_\zeta} + \dots \tag{6.14}$$

Now we can easily generalize the result to the polynomials including the four-dimensional class

$$\begin{aligned} &\overline{q}_{X,k}(\Sigma^{d-2r}(pt)^r)_{\mathbf{B}} \\ &\quad = \frac{1}{2}\sum_{\ell=0}^{k}(-1)^{k-\ell}2^{\ell-3r}\frac{(d-2r)!\,r!}{(d-2\ell-2r)!\,\ell!}\sum_{\substack{\zeta_i\cdot H<0,\\ \zeta_i\cdot\zeta_i=-k+\ell}}(\zeta_i\cdot\Sigma)^{d-2\ell-2r}\,q^{\ell} + \cdots, \end{aligned} \tag{6.15}$$

where $r = 0, 1, .., [d/2] - \ell$. In other words,

$$\overline{\Gamma}^{k,r}_X(C)_{\mathbf{B}} = \frac{1}{2}\sum_{\ell=0}^{k}(-1)^{k-\ell}2^{\ell-3r}\frac{(d-2r)!\,r!}{(d-2\ell-2r)!\,\ell!}\sum_{\substack{\zeta_i\cdot H<0,\\ \zeta_i\cdot\zeta_i=-k+\ell}}(\zeta_i)^{d-2\ell-2r}q^{\ell} + \cdots. \tag{6.16}$$

Thus the transition formula is given by

$$\overline{\Gamma}^{k,r}_X(C_+) - \overline{\Gamma}^{k,r}_X(C_-) = (-1)^{k-\ell_\zeta} 2^{\ell_\zeta-3r} \frac{(d-2r)!\,r!}{(d-2\ell_\zeta-2r)!\,\ell_\zeta!} (\zeta)^{d-2\ell_\zeta-2r} q^{\ell_\zeta} + \cdots. \tag{6.17}$$

This formula has been obtained by Kotschick and Morgan [7] for $r = 0$, by Friedman and Qin [33] for $r = 1$ and by Ellingsrud and Göttsche [34] for general r. The paper [33] has some explicit results beyond the leading term. In the paper [34], up to the leading 3-terms were calculated for general r.

6.2. **The variation of the moduli space**

It is well-known that the image of moment map is a convex cone in a positive Weyl chamber of the Lie algebra and the critical values of the moment map define a system of walls in the convex cone [60]. It is also known that the symplectic quotients undergo a specific birational transformations closely related to the variation of GIT quotients as the values of moment map cross the wall[61][62][63]. The walls are determined by the critical points of the moment map. Of course, all these are rigorous for finite dimensional compact manifold with compact group actions.

Formally speaking, our problem is an infinite dimensional analogue of the variation of the symplectic quotients. The moduli space of ASD connection can be identified with the symplectic quotients $\mathfrak{m}(0)^{-1}/\mathcal{G}$. An interesting fact is that the chamber structures in the positive cone of the manifold and those in the convex cone sitting on image of the moment map are determined by the same data. Our results imply that the two chamber structures are isomorphic! Naively speaking, this suggests that the moduli space of ASD connections undergoes certain birational transformations if the metric cross that walls. This picture also coincides with our general strategy for deriving the transition formula. Losing a higher critical point ζ and getting another higher critical point $-\zeta$ is analogous to the blown-down and successive blown-up.

In the algebro-geometrical method, one constructs the moduli space of H-stable bundles over algebraic surface and studies the variation of the moduli space under the changes of the polarizations [64][65]. One of the advantage of the algebro-geometrical approach is that it has a natural way of the compactification of the moduli space. Since the moduli space of H-semi-stable bundles contains the moduli space of the H-stable bundles as a Zariski open subset, it gives a natural compactification. It is well-known that the diffeomorphism class of the moduli space depends on the chamber structure in the ample (Kähler) cone. Recently several papers on the variation of the moduli space of H-semi-stable bundle under the changes of the polarization of the ample class appeared [34][35][33]. They show that the moduli space undergoes specific birational transformation similar to the variation of geometrical invariant theory (GIT) quotients studied by Thaddeus [66][62] and Dolgachev-Hu [63]. The variation of GIT quotients was also studied independently by Witten in the context of two-dimensional supersymmetric theory and the quantum cohomology rings[67]. We note that Witten's picture is quite similar to our approach.

It is not yet clear if a natural compactification of the moduli space can be obtained using some path integral methods. However, our result is sufficient to predict the general form of the Donaldson invariants as (6.1)(6.2)(6.3). If we consider when the variation of the Donaldson invariants get contribution only from $\overline{q}_{X,k,C(X)}(\Sigma^{d-2r}(pt)^r)_{\mathbf{B}}$, the transition formula of the Donaldson invariants is sufficient to determine $\overline{q}_{X,k,C(X)}(\Sigma^{d-2r}(pt)^r)_{\mathbf{B}}$ as one can see from the relation between (6.17) to (6.16) and (6.15).

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